بسم الله الرحمن الرحيم

Vibration

Fayoum University

Faculty of Engineering
Mechanical Engineering Dept.

Lecture (6)
on
The Vibrations of Systems Having Two Degree of Freedom

By

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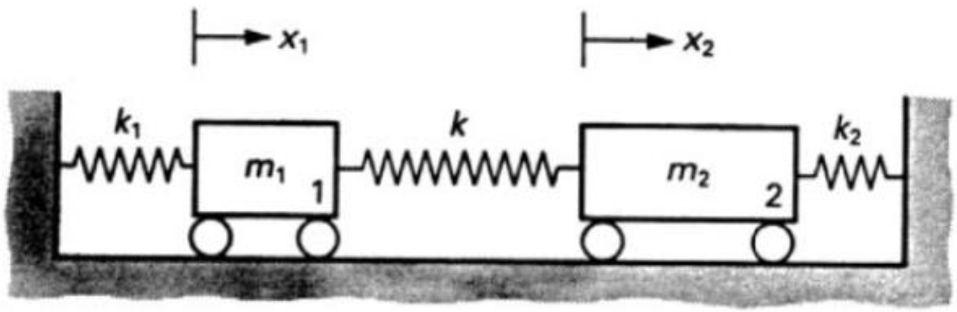
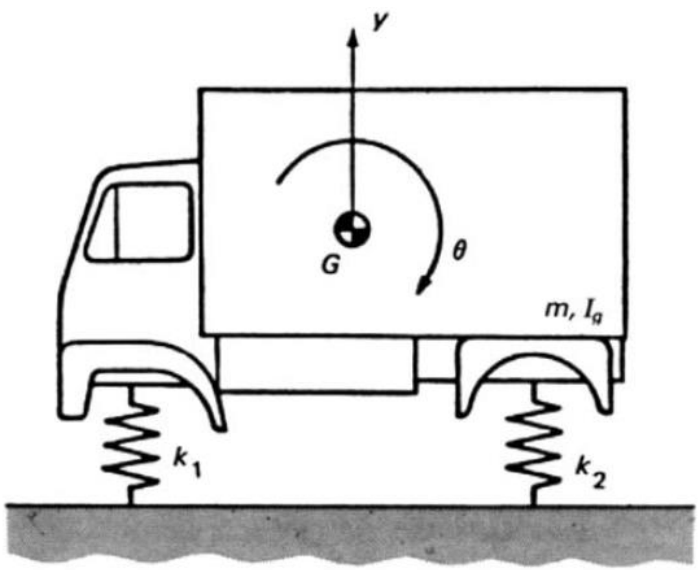
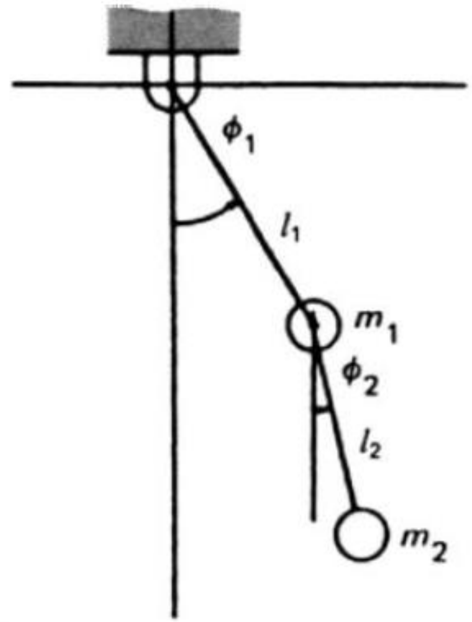
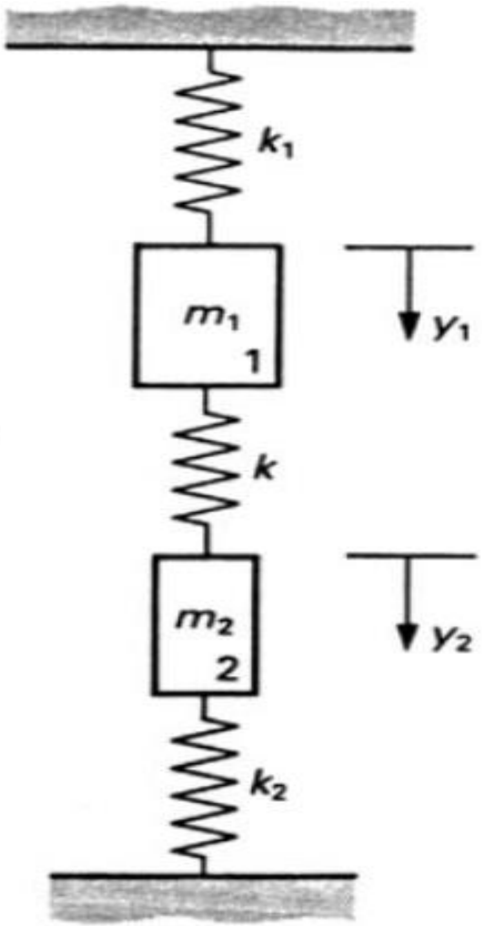
Systems Having Two Degree of Freedom



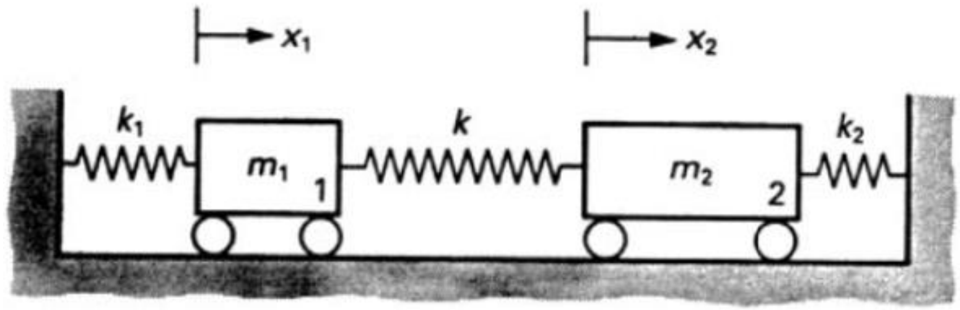






Free Vibration of an Undamped System

If $x_1 > x_2$ the FBDs are



The equations of motion are therefore,

$$m_1\ddot{x}_1 = -k_1x_1 - k(x_1 - x_2) \quad \text{for body 1}$$

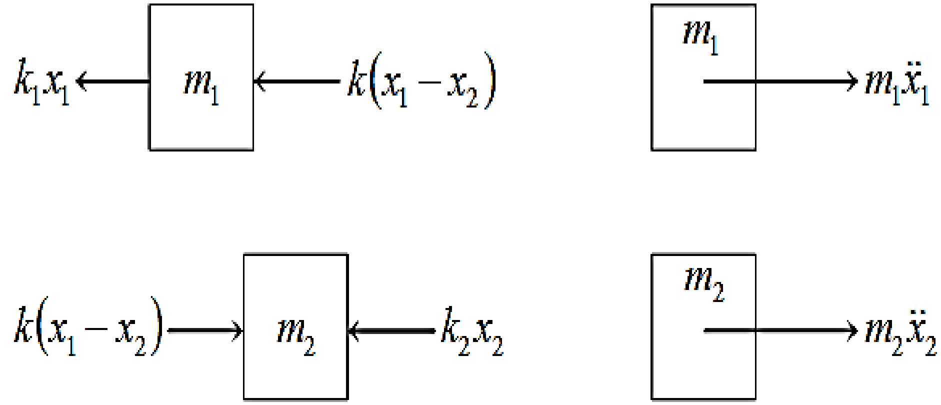
and

$$m_2\ddot{x}_2 = k(x_1 - x_2) - k_2x_2 \quad \text{for body 2.}$$

The same equations are obtained if $x_1 < x_2$ is assumed because the direction of the central spring force is then reversed.

Above equations can be solved for the natural frequencies and corresponding mode shapes by assuming a solution of the form

$$x_1 = A_1 \sin(\omega t + \phi) \text{ and } x_2 = A_2 \sin(\omega t + \phi)$$



Applied forces. effective forces.

Free Vibration of an Undamped System

Substituting these solutions into the equations of motion gives

$$-m_1 A_1 \omega^2 \sin(\omega t+\phi) = -k_1 A_1 \sin(\omega t+\phi) - k(A_1 - A_2)\sin(\omega t+\phi)$$

and

$$-m_2 A_2 \omega^2 \sin(\omega t+\phi) = k(A_1 - A_2)\sin(\omega t+\phi) - k_2 A_2 \sin(\omega t+\phi)$$

Since these solutions are true for all values of t,

$$A_1(k + k_1 - m_1\omega^2) + A_2(-k) = 0$$

and

$$A_1(-k) + A_2(k_2 + k - m_2\omega^2) = 0$$

A_1 and A_2 can be eliminated by writing

Free Vibration of an Undamped System

$$\begin{vmatrix} k + k_1 - m_1\omega^2 & -k \\ -k & k + k_2 - m_2\omega^2 \end{vmatrix} = 0$$

This is the characteristic or frequency equation. Alternatively, we may write

$$\left.\begin{array}{l} A_1 / A_2 = k / (k + k_1 - m_1\omega^2) \quad \textit{from } (4.3) \\ \textit{and} \\ A_1 / A_2 = (k_2 + k - m_2\omega^2) / k \quad \textit{from } (4.4) \end{array}\right\}$$

Thus

$$k/(k + k_1 - m_1\omega^2) = (k_2 + k - m_2\omega^2)/k$$

and

$$(k + k_1 - m_1\omega^2)(k_2 + k - m_2\omega^2) - k^2 = 0$$

Free Vibration of an Undamped System

Consider the case when $k_1 = k_2 = k$, and $m_1 = m_2 = m$. The frequency equation is $(2k - m\omega^2)^2 - k^2 = 0$; that is,

$$m^2\omega^4 - 4mk\omega^2 + 3k^2 = 0 \quad \text{or} \quad (m\omega^2 - k)(m\omega^2 - 3k) = 0$$

Therefore, either $m\omega^2 - k = 0$, or $m\omega^2 - 3k = 0$

Thus

$$\omega_1 = \sqrt{\frac{k}{m}}\ rad/s \quad \text{and} \quad \omega_2 = \sqrt{\frac{3k}{m}}\ rad/s$$

If

$$\omega = \sqrt{\frac{k}{m}}\ rad/s,\ (A_1/A_2)_{\omega=\sqrt{\frac{k}{m}}} = +1$$

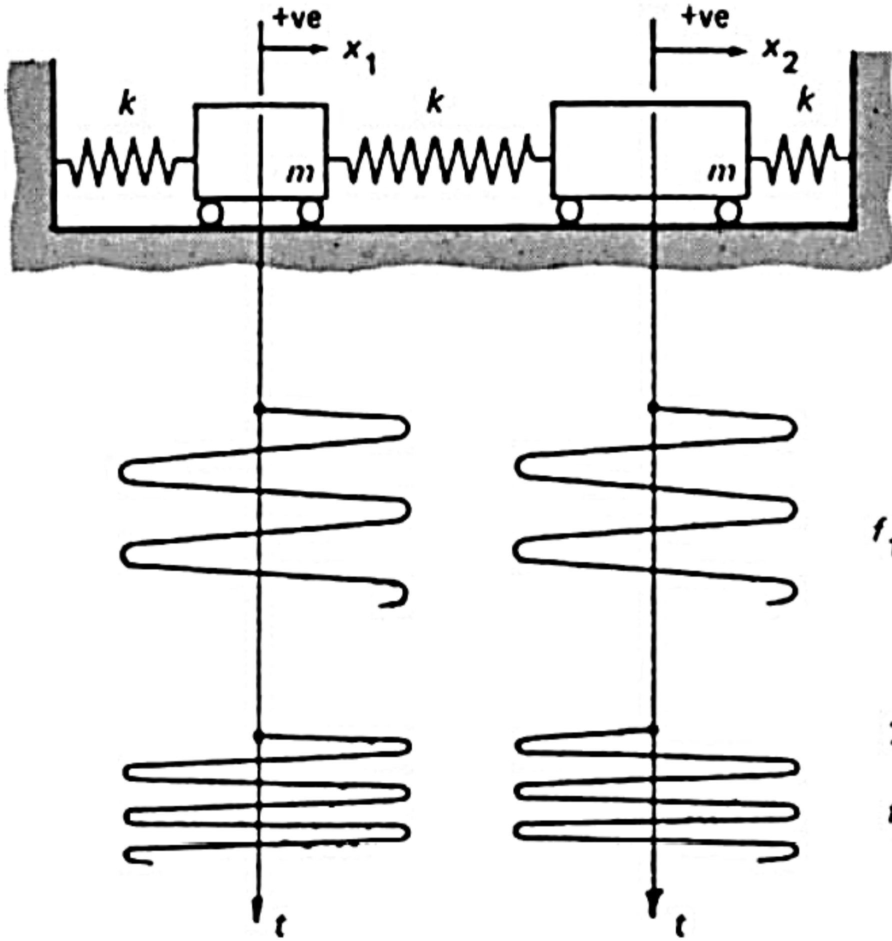
and if

$$\omega = \sqrt{\frac{3k}{m}}\ rad/s,\ (A_1/A_2)_{\omega=\sqrt{\frac{3k}{m}}} = -1$$

Free Vibration of an Undamped System

This gives the mode shapes corresponding to the frequencies ω_1 and ω_2. Thus, the first mode of free vibration occurs at a frequency $(1/2\pi)\sqrt{k/m}\,Hz$ and $(A_1/A_2)^I = 1$, that is, the bodies move in phase with each other and with the same amplitude as if connected by a rigid link, Figure (4.3). The second mode of free vibration occurs at a frequency $(1/2\pi)\sqrt{3k/m}\,Hz$ and $(A_1/A_2)^{II} = -1$, that is, the bodies move exactly out of phase with each other, but with the same amplitude, see Figure (4.3).

Free Vibration of an Undamped System



1st mode:

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\ \text{Hz}. \quad \frac{A_1}{A_2} = +1.$$

2nd mode:

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{3k}{m}}\ \text{Hz}. \quad \frac{A_1}{A_2} = -1.$$

Free Motion

The two modes of vibration can be written

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{\mathrm{I}} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}^{\mathrm{I}} \sin(\omega_1 t + \phi_1), \quad \text{and}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{\mathrm{II}} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}^{\mathrm{II}} \sin(\omega_2 t + \phi_2),$$

where the ratio A_1/A_2 is specified for each mode.

Since each solution satisfies the equation of motion, the general solution is

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}^{\mathrm{I}} \sin(\omega_1 t + \phi_1) + \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}^{\mathrm{II}} \sin(\omega_2 t + \phi_2),$$

where A_1, A_2, ϕ_1, ϕ_2 are found from the initial conditions.

For example, for the system considered above, if one body is displaced a distance X and released,

$$x_1(0) = X \quad \text{and} \quad x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$$

where $x_1(0)$ means the value of x_1 when $t = 0$, and similarly for $x_2(0)$, $\dot{x}_1(0)$ and $\dot{x}_2(0)$.

Free Motion

Remembering that in this system $\omega_1 = \sqrt{k/m}$, $\omega_2 = \sqrt{3k/m}$, and

$$\left(\frac{A_1}{A_2}\right)_{\omega_1} = +1 \quad \text{and} \quad \left(\frac{A_1}{A_2}\right)_{\omega_2} = -1$$

we can write

$$x_1 = \sin\left(\sqrt{(k/m)}t + \phi_1\right) + \sin\left(\sqrt{(3k/m)}t + \phi_2\right), \quad \text{and}$$

$$x_2 = \sin\left(\sqrt{(k/m)}t + \phi_1\right) - \sin\left(\sqrt{(3k/m)}t + \phi_2\right)$$

Substituting the initial conditions $x_1(0) = X$ and $x_2(0) = 0$ gives

$$X = \sin\phi_1 + \sin\phi_2, \quad \text{and}$$

$$0 = \sin\phi_1 - \sin\phi_2$$

that is,

$$\sin\phi_1 = \sin\phi_2 = X/2$$

The remaining conditions give $\cos\phi_1 = \cos\phi_2 = 0$.

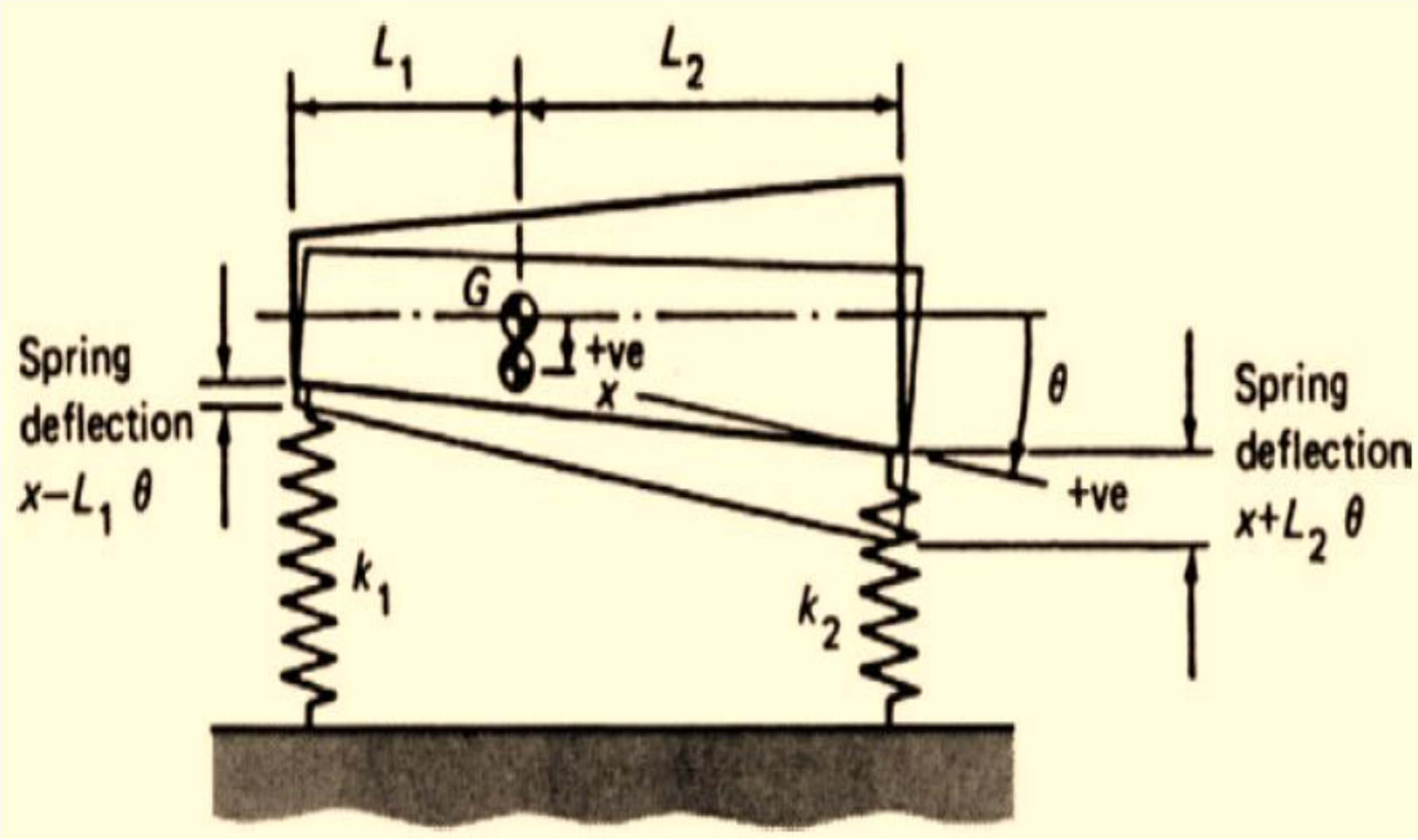
Hence

$$x_1 = (X/2)\cos\sqrt{(k/m)}\,t + (X/2)\cos\sqrt{(3k/m)}t,$$

and $x_2 = (X/2)\cos\sqrt{(k/m)}\,t - (X/2)\cos\sqrt{(3k/m)}t$

That is, both natural frequencies are excited and the motion of each body has two harmonic components.

Coordinate Coupling



Coordinate Coupling

For small amplitudes of oscillation (so that $\sin\theta \approx \theta$) the equations of motion are

$$m\ddot{x} = -k_1(x - L_1\theta) - k_2(x + L_2\theta)\,,$$

and

$$I\ddot{\theta} = k_1(x - L_1\theta)L_1 - k_2(x + L_2\theta)L_2$$

that is,

$$m\ddot{x} + (k_1 + k_2)x - (k_1L_1 - k_2L_2)\theta = 0\,,$$

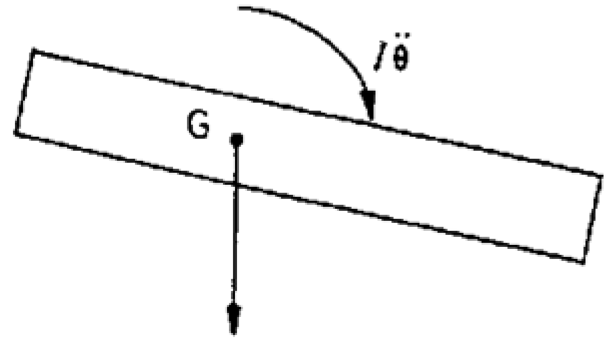
and

$$I\ddot{\theta} - (k_1L_1 - k_2L_2)x + (k_1L_1^2 + k_2L_2^2)\theta = 0$$

(a)

Applied forces,

(b)



effective forces.

Coordinate Coupling

It will be noticed that these equations can be uncoupled by making $k_1L_1 = k_2L_2$; if this is arranged, translation (x motion) and rotation (θ motion) can take place independently. Otherwise translation and rotation occur simultaneously.

Assuming $x_1 = A_1 \sin(\omega t + \phi)$ and $\theta = A_2 \sin(\omega t + \phi)$, substituting into the equations of motion gives

$$-m\omega^2 A_1 + (k_1 + k_2)A_1 - (k_1L_1 - k_2L_2)A_2 = 0,$$

and

$$-I\omega^2 A_2 - (k_1L_1 - k_2L_2)A_1 + (k_1L_1^2 + k_2L_2^2)A_2 = 0$$

Coordinate Coupling

that is,

$$A_1\left(k_1 + k_2 - m\omega^2\right) + A_2\left(-\left(k_1L_1 - k_2L_2\right)\right) = 0$$

and

$$A_1\left(-\left(k_1L_1 - k_2L_2\right)\right) + A_2\left(k_1L_1^2 + k_2L_2^2 - I\omega^2\right) = 0$$

Hence the frequency equation is

$$\begin{vmatrix} k_1 + k_2 - m\omega^2 & -\left(k_1L_1 - k_2L_2\right) \\ -\left(k_1L_1 - k_2L_2\right) & k_1L_1^2 + k_2L_2^2 - I\omega^2 \end{vmatrix} = 0$$

For each natural frequency, there is a corresponding mode shape, given by A_1/A_2.

Example 1: Coordinate Coupling

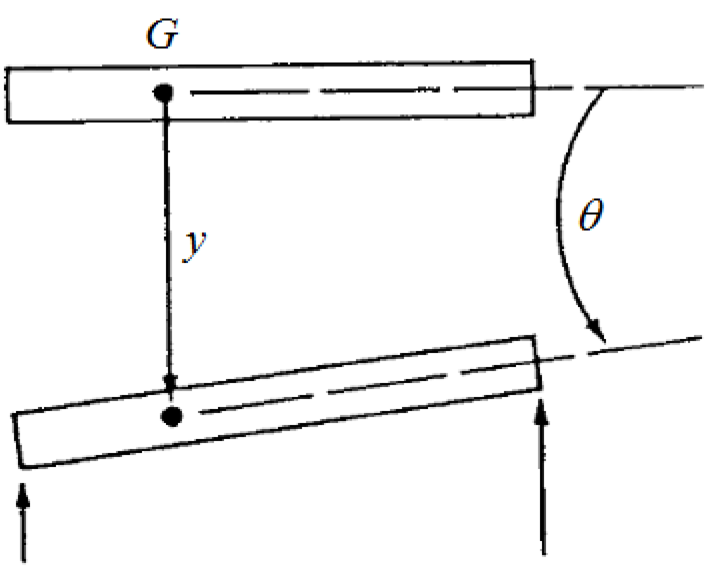
When transported, a space vehicle is supported in a horizontal position by two springs, as shown in Figure (E4.2). The vehicle can be considered to be a rigid body of mass m and radius of gyration h about an axis normal to the plane of the figure through the mass center G. The rear support has a stiffness k_1 and is at a distance a from G while the front support has a stiffness k_2 and is at a distance b from G. The only motions possible for the vehicle are vertical translation and rotation in the vertical plane.

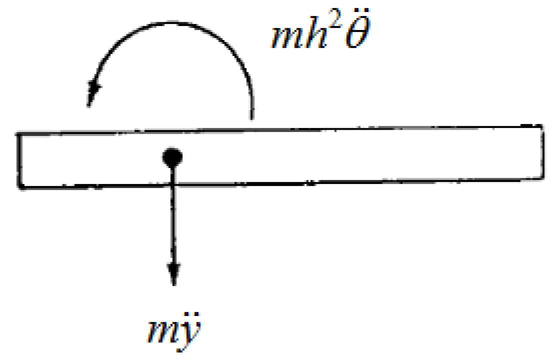
Write the equations of small amplitude motion of the vehicle and obtain the frequency equation in terms of the given parameters.

Given that $k_1 a = k_2 b$, determine the natural frequencies of the free vibrations of the vehicle and sketch the corresponding modes of vibration. Also state or sketch the modes of vibration if $k_1 a \neq k_2 b$.

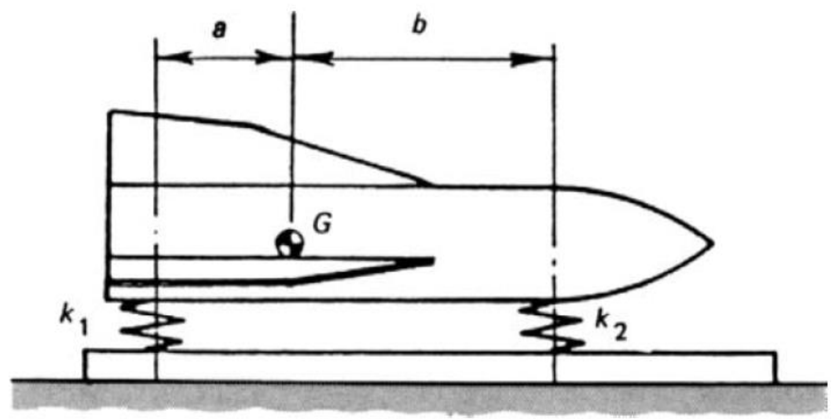
Example 1: Coordinate Coupling

The FBDs are as below:





The equations of motion are



Example 1: Coordinate Coupling

$$k_1(y+a\theta)+k_2(y-b\theta)=-m\ddot{y},$$

and

$$k_1a(y+a\theta)-k_2b(y-b\theta)=-mh^2\ddot{\theta}$$

Assuming

$$y=Y\sin\omega t \quad \text{and} \quad \theta=\Theta\sin\omega t$$

these give

$$Y(k_1+k_2-m\omega^2)+\Theta(k_1a-k_2b)=0,$$

and

$$Y(k_1a-k_2b)+\Theta(k_1a^2+k_2b^2-mh^2\omega^2)=0$$

The frequency equation is, therefore,

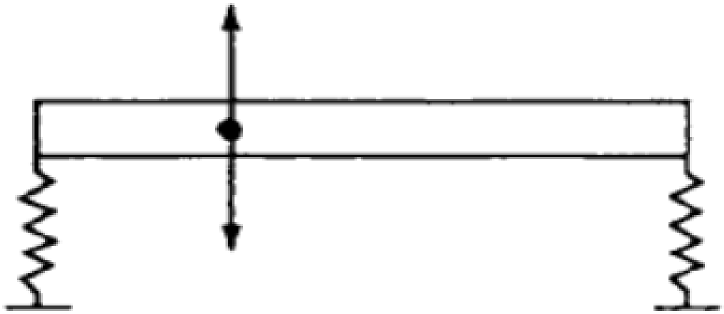
$$(k_1+k_2-m\omega^2)(k_1a^2+k_2b^2-mh^2\omega^2)-(k_1a-k_2b)=0$$

If $k_1a = k_2b$, motion is uncoupled so

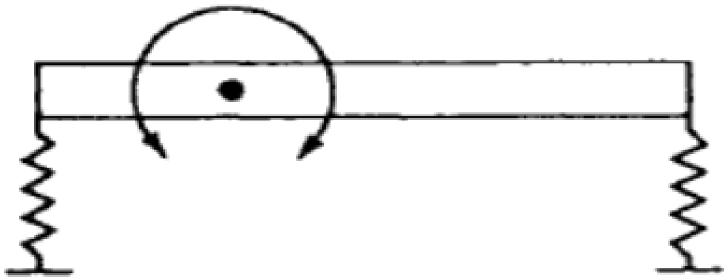
$$\omega_1=\sqrt{\left(\frac{k_1+k_2}{m}\right)}\ rad/s \quad \text{and} \quad \omega_2=\sqrt{\left(\frac{k_1a^2+k_2b^2}{mh^2}\right)}\ rad/s$$

Example 1: Coordinate Coupling

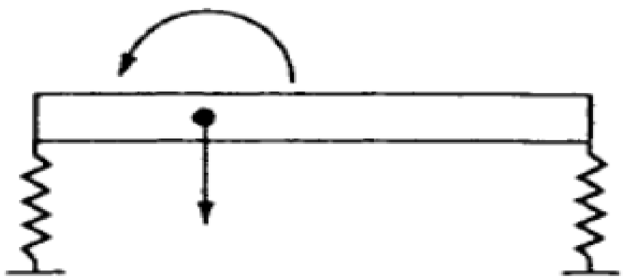
ω_1 is the frequency of a bouncing or translation mode (no rotation):

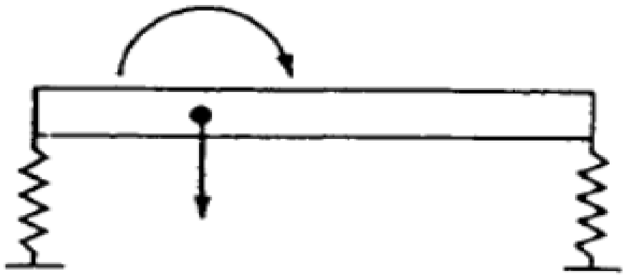


ω_2 is the frequency of a rotation mode (no bounce):



If $k_1 a \neq k_2 b$, the modes are coupled:

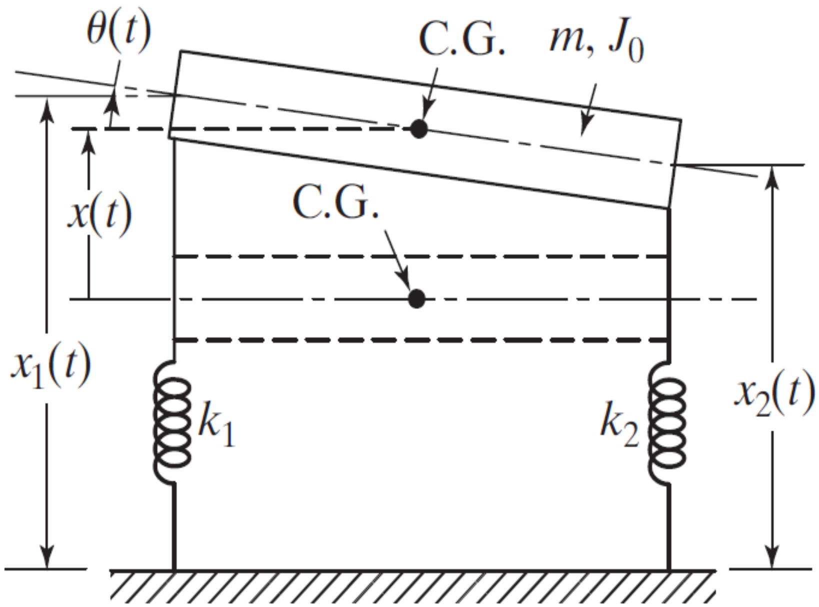





QUIZ

Quiz

The motor-pump system shown in Figure (P4.5a) is modeled as a rigid bar of mass $m = 50$ kg and mass moment of inertia $J_0 = 100$ kg.m^2. The foundation of the system can be replaced by two springs of stiffness $k_1 = 500$ N/m and $k_2 = 200$ N/m as shown in Figure (P4.5b). Determine the natural frequencies of the system.



Thank you