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Differential Subordination and Superordination Results for Higher-Order Derivatives of p-Valent Functions Involving a Generalized Differential Operator

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Abstract. The purpose of this paper is to obtain some subordination, superordination and sandwich results for higher-order derivatives of $p-$valent functions involving a generalized differential operator. Some of our results generalize previously known results.

Keywords: Analytic function; Hadamard product; Differential subordination; Superordination; Sandwich theorems; Linear operator.

1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, p]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} ... \ (a \in \mathbb{C};\ p \in \mathbb{N} = \{1, 2, ...\}).$$

For simplicity $H[a] = H[a, 1]$. Also, let $\mathcal{A}(p)$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}), \tag{1.1}$$

which are $p-$valent in U. We write $\mathcal{A}(1) = \mathcal{A}$.

If $f,\ g \in H(U)$, we say that f is subordinate to g or g is superordinate to f, written $f(z) \prec g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)),\ z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g. [14], [21] and [22]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h be univalent in U. If β is analytic in U and satisfies the first order differential subordination:

$$\phi\left(\beta(z), z\beta'(z); z\right) \prec h(z), \tag{1.2}$$

then β is a solution of the differential subordination (1.2). The univalent function q is called a dominant of the solutions of the differential subordination (1.2) if $\beta(z) \prec q(z)$ for all β satisfying (1.2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.2) is called the best dominant. If β and ϕ are univalent functions in U and if satisfies first order differential superordination:

$$h(z) \prec \phi\left(\beta(z), z\beta'(z); z\right), \tag{1.3}$$

then β is a solution of the differential superordination (1.3). An analytic function q is called a subordinant of the solutions of the differential superordination (1.3) if $q(z) \prec \beta(z)$ for all β satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfing $q(z) \prec \tilde{q}(z)$ for all subordinants of (1.3) is called the best subordinant.

Using the results of Miller and Mocanu [22], Bulboaca [13] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [14]. Ali et al. [1], have used the results of Bulboaca [13] to obtain sufficient conditions for normalized analytic functions $f \in \mathcal{A}$ to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [27] obtained a sufficient condition for starlikeness of $f \in \mathcal{A}$ in terms of the quantity $\dfrac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [26] obtained sufficient conditions for the normalized analytic function $f \in \mathcal{A}$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

$$q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

For functions $f \in \mathcal{A}(p)$ given by (1.1) and $g \in \mathcal{A}(p)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \quad (p \in \mathbb{N}), \tag{1.4}$$

the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.5}$$

Upon differentiating both sides of (1.5) $j-$times with respect to z, we have

$$(f * g)^{(j)}(z) = \delta(p; j) z^{p-j} + \sum_{k=p+1}^{\infty} \delta(k; j) a_k b_k z^{k-j}, \tag{1.6}$$

where

$$\delta(p; j) = \frac{p!}{(p-j)!} \quad (p > j; p \in \mathbb{N}; j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.7}$$

For functions $f, g \in \mathcal{A}(p)$, we define the linear operator $D_{\lambda,p}^{n}(f * g)^{(j)} : \mathcal{A}(p) \to \mathcal{A}(p)$ by:

$$\begin{aligned}
D_{\lambda,p}^{0}(f * g)^{(j)}(z) &= (f * g)^{(j)}(z), \\
D_{\lambda,p}^{1}(f * g)^{(j)}(z) &= D_{\lambda,p}(f * g)^{(j)}(z) \\
&= (1-\lambda)(f * g)^{(j)}(z) + \frac{\lambda}{p-j} z \left((f * g)^{(j)}\right)'(z) \\
&= \delta(p; j) z^{p-j} + \sum_{k=p+1}^{\infty} \left(\frac{p-j+\lambda(k-p)}{p-j}\right) \delta(k; j) a_k b_k z^{k-j}, \\
D_{\lambda,p}^{2}(f * g)^{(j)}(z) &= D\left(D_{p}^{1}(f * g)^{(j)}(z)\right) \\
&= \delta(p; j) z^{p-j} + \sum_{k=p+1}^{\infty} \left(\frac{p-j+\lambda(k-p)}{p-j}\right)^{2} \delta(k; j) a_k b_k z^{k-j},
\end{aligned}$$

and (in general)

$$\begin{aligned}
D_{\lambda,p}^{n}(f * g)^{(j)}(z) &= D(D_{p}^{n-1}(f * g)^{(j)}(z)) \\
&= \delta(p; j) z^{p-j} + \sum_{k=p+1}^{\infty} \left(\frac{p-j+\lambda(k-p)}{p-j}\right)^{n} \delta(k; j) a_k b_k z^{k-j}, \\
&\quad (\lambda \geq 0; p > j; p \in \mathbb{N}; j, n \in \mathbb{N}_0; z \in U).
\end{aligned} \tag{1.8}$$

From (1.8), we can easily deduce that

$$\frac{\lambda z}{p-j}\left(D_{\lambda,p}^{n}(f * g)^{(j)}(z)\right)' = D_{\lambda,p}^{n+1}(f * g)^{(j)}(z) - (1-\lambda) D_{\lambda,p}^{n}(f * g)^{(j)}(z)$$
$$(\lambda > 0; p > j; p \in \mathbb{N}; n, j \in \mathbb{N}_0; z \in U). \tag{1.9}$$

We observe that the linear operator $D_{\lambda,p}^{n}(f*g)^{(j)}(z)$ reduces to several interesting many other linear operators considered earlier for different choices of j, n, λ and the function g:

(i) For $j = 0, D_{\lambda,p}^{n}(f*g)^{(0)}(z) = D_{\lambda,p}^{n}(f*g)(z)$, where the operator $D_{\lambda,p}^{n}(f*g)$ $(\lambda \geq 0,\ p \in \mathbb{N}, n \in \mathbb{N}_0)$ was introduced and studied by Selvaraj et al. [25] (see also [12]) and $D_{\lambda,1}^{n}(f*g)(z) = D_{\lambda}^{n}(f*g)(z)$, where the operator $D_{\lambda}^{n}(f*g)$ was introduced by Aouf and Mostafa [6] (see also [8]);

(ii) For

$$g(z) = \frac{z^p}{1-z} \quad (p \in \mathbb{N}; z \in U\ ) \tag{1.10}$$

we have $D_{\lambda,p}^{n}(f*g)^{(j)}(z) = D_{\lambda,p}^{n}f^{(j)}(z)$, $D_{\lambda,p}^{n}f^{(0)}(z) = D_{\lambda,p}^{n}f(z)$, where the operator $D_{\lambda,p}^{n}$ is the $p-$valent Al-Oboudi operator which was introduced by El-Ashwah and Aouf [17], $D_{1,p}^{n}f^{(j)}(z) = D_{p}^{n}f^{(j)}(z)$, where the operator $D_{p}^{n}f^{(j)}$ $(p > j, p \in \mathbb{N}, n, j \in \mathbb{N}_0)$ was introduced and studied by Aouf [3,4] (see also [7]) and $D_{1,p}^{n}f^{(0)}(z) = D_{p}^{n}f(z)$, where the operator D_{p}^{n} is the $p-$valent Sălăgean operator which was introduced and studied by Kamali and Orhan [18] (see also [5], [10] and [11]);

(iii) For

$$g(z) = z^p + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p}...(\alpha_q)_{k-p}}{(\beta_1)_{k-p}...(\beta_s)_{k-p}} \frac{z^k}{(1)_{k-p}} \qquad (z \in U), \tag{1.11}$$

(for complex parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ $(\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}$, $j = 1, ..., s)$; $q \leq s+1; p \in \mathbb{N};\ q,\ s \in \mathbb{N}_0)$ where $(\nu)_k$ is the Pochhammer symbol defined in terms to the Gamma function Γ, by

$$(\nu)_k = \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1, & (k=0), \\ \nu(\nu+1)(\nu+2)...(\nu+k-1), & (k \in \mathbb{N}), \end{cases}$$

we have $D_{\lambda,p}^{n}(f*g)^{(j)}(z) = D_{\lambda,p}^{n}(H_{p,q,s}(\alpha_1)f)^{(j)}(z)$ and $D_{\lambda,p}^{0}(f*g)^{(0)}(z) = H_{p,q,s}(\alpha_1)f(z)$, where the operator $H_{p,q,s}(\alpha_1) = H_{p,q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)$ is the Dziok-Srivastava operator which was introduced and studied by Dziok and Srivastava [16] and contains in turn many interesting operators;

(iv) For

$$g(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+l+\alpha(k-p)}{p+l} \right)^m z^k \tag{1.12}$$
$$(\alpha \geq 0;\ l \geq 0;\ p \in \mathbb{N};\ m \in \mathbb{N}_0; z \in U\ ),$$

we have $D_{\lambda,p}^{n}(f*g)^{(j)}(z) = D_{\lambda,p}^{n}(I_p(m, \alpha, l)f)^{(j)}(z)$, and $D_{\lambda,p}^{0}(f*g)^{(0)}(z) = I_p(m, \alpha, l)f(z)$, where the operator $I_p(m, \alpha, l)$ was introduced and studied by Cătas [15] which contains in turn many interesting operators such as, $I_p(m, 1, l) = I_p(m, l)$, where the operator $I_p(m, l)$ was investigated by Kumar et al. [19];

(v) For

$$g(z) = z^p + \frac{\Gamma(p+\alpha+\beta)}{\Gamma(p+\beta)} \sum_{k=p+1}^{\infty} \frac{\Gamma(k+\beta)}{\Gamma(k+\alpha+\beta)} z^k \tag{1.13}$$
$$(\ \alpha \geq 0;\ p \in \mathbb{N};\ \beta > -1; z \in U\ )$$

we have $D_{\lambda,p}^{n}(f*g)^{(j)}(z) = D_{\lambda,p}^{n}\left(Q_{\beta,p}^{\alpha} f\right)^{(j)}(z)$ and $D_{\lambda,p}^{0}(f*g)^{(0)}(z) = Q_{\beta,p}^{\alpha}$ $f(z)$, where the operator $Q_{\beta,p}^{\alpha}$ was introduced and studied by Liu and Owa [20] (see also [9]);

(vi) For $j = 0$ and g of the form (1.11) with $p = 1$, we have $D_{\lambda,1}^{n}(f*g)(z) = D_{\lambda}^{n}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)(z)$, where the operator $D_{\lambda}^{n}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)$ was introduced and studied by Selvaraj and Karthikeyan [24];

(vii) For $j = 0$, $p = 1$ and

$$g(z) = z + \sum_{k=2}^{\infty} \left[\frac{\Gamma(k+1)\Gamma(2-m)}{\Gamma(k+1-m)}\right]^n z^k, \tag{1.14}$$

where $n \in \mathbb{N}_0; 0 \leq\ m < 1; z \in U$, we have $D_{\lambda,1}^{n}(f*g)(z) = D_{\lambda}^{n,m} f(z)$, where the operator $D_{\lambda}^{n,m}$ was introduced and studied by Al-Oboudi and Al-Amoudi [2].

In this paper, we will derive several subordination, superordination and sandwich results involving the operator $D_{\lambda,p}^{n}(f*g)^{(j)}$.

2. Definitions and Preliminaries

In order to prove our results, we need the following definition and lemmas.

Definition 2.1. [22] *Denote by Q, the set of all functions f that are analytic and injective on $\overline{U}\backslash E(f)$, where $E(f) = \left\{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\right\}$, and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U\backslash E(f)$.*

Lemma 2.2. [26] *Let q be univalent function in U with $q(0) = 1$. Let $\gamma_i \in \mathbb{C}(i = 1, 2)$, $\gamma_2 \neq 0$, further assume that*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\Re\left(\frac{\gamma_1}{\gamma_2}\right)\right\}. \tag{2.1}$$

If β is analytic function in U, and

$$\gamma_1 \beta(z) + \gamma_2 z\beta'(z) \prec \gamma_1 q(z) + \gamma_2 zq'(z),$$

then $\beta \prec q$ and q is the best dominant.

Lemma 2.3. [26] *Let q be convex univalent function in U, $q(0)=1$. Let $\gamma_i \in \mathbb{C}(i=1,2)$, $\gamma_2 \neq 0$ and $\Re\left(\frac{\gamma_1}{\gamma_2}\right) > 0$. If $\beta \in H[q(0),1] \cap Q$, $\gamma_1\beta(z)+\gamma_2 z\beta'(z)$ is univalent in U and*

$$\gamma_1 q(z)+\gamma_2 zq'(z) \prec \gamma_1\beta(z)+\gamma_2 z\beta'(z), \tag{2.2}$$

then $q \prec \beta$ and q is the best subordinant.

3. Subordination Resuts

Unless otherwise mentioned, we assume throughout this paper that $\gamma \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$, $\lambda > 0$, $p > j$, $p \in \mathbb{N}$, $n, j \in \mathbb{N}_0$ and $\delta(p;j)$ is given by (1.7).

Theorem 3.1. *Let $q(z)$ be univalent in U with $q(0)=1$. Further, assume that*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\Re\left(\frac{1}{\gamma}\right)\right\}. \tag{3.1}$$

If $f \in \mathcal{A}(p)$ satisfies the following subordination condition:

$$\left[1+\gamma\left(\frac{p-j}{\lambda}\right)\right]\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^2}$$
$$+\gamma\left(\frac{p-j}{\lambda}\right)\delta(p;j)\,z^{p-j}\left\{\frac{D_{\lambda,p}^{n+2}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^2}-\frac{2\left[D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)\right]^2}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^3}\right\}$$
$$\prec q(z)+\gamma zq'(z),$$

then

$$\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^2} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define a function $\varrho(z)$ by

$$\varrho(z)=\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^2} \quad (z \in U). \tag{3.3}$$

Then the function ϱ is analytic in U and $\varrho(0)=1$. Therefore, differentiating (3.3) logarithmically with respect to z and using the identity (1.9) in the resulting equation, we have

$$\left[1+\gamma\left(\frac{p-j}{\lambda}\right)\right]\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}}$$
$$+\gamma\left(\frac{p-j}{\lambda}\right)\delta(p;j)\,z^{p-j}\left\{\frac{D_{\lambda,p}^{n+2}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}}-\frac{2\left[D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)\right]^{2}}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{3}}\right\}$$
$$=\varrho(z)+\gamma z\varrho'(z),$$

that is, $\varrho(z)+\gamma z\varrho'(z)\prec q(z)+\gamma zq'(z)$. Therefore, Theorem 3.1 now follows by applying Lemma 2.2. ■

Putting $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 3.1, it is easy to check that the assumption (3.1) holds whenever $-1\le B<A\le 1$, hence we obtain the following corollary.

Corollary 3.2. *Let $-1\le B<A\le 1$ and assume that*

$$\Re\left\{\frac{1-Bz}{1+Bz}\right\}>\max\left\{0,-\Re\left(\frac{1}{\gamma}\right)\right\}.$$

If $f\in\mathcal{A}(p)$ satisfy the following subordination condition

$$\left[1+\gamma\left(\frac{p-j}{\lambda}\right)\right]\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}}$$
$$+\gamma\left(\frac{p-j}{\lambda}\right)\delta(p;j)\,z^{p-j}\left\{\frac{D_{\lambda,p}^{n+2}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}}-\frac{2\left[D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)\right]^{2}}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{3}}\right\}$$
$$\prec\frac{1+Az}{1+Bz}+\gamma\frac{(A-B)\,z}{(1+Bz)^{2}},$$

then

$$\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}}\prec\frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominan.

Taking $g=\frac{z^{p}}{1-z}$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.3. *Let q be univalent in U with $q(0)=1$ and assume that* (3.1) *holds.*

If $f \in \mathcal{A}(p)$ satisfies the following subordination condition:

$$\begin{aligned}&\left[1+\gamma\left(\frac{p-j}{\lambda}\right)\right]\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}f^{(j)}(z)}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^{2}}\\&+\gamma\left(\frac{p-j}{\lambda}\right)\delta(p;j)\,z^{p-j}\left\{\frac{D_{\lambda,p}^{n+2}f^{(j)}(z)}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^{2}}-\frac{2\left[D_{\lambda,p}^{n+1}f^{(j)}(z)\right]^{2}}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^{3}}\right\}\\&\prec q(z)+\gamma zq'(z),\end{aligned}$$

then

$$\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}f^{(j)}(z)}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^{2}}\prec q(z)$$

and $q(z)$ is the best dominant.

Remark 3.4. (i) Taking $\lambda = 1$ in Corollary 3.3, we obtain the result obtained by Aouf and Seoudy [7, Theorem 1];

(ii) Taking $p = 1$, $j = 0$ and $g = \frac{z}{1-z}$ in Theorem 3.1, we obtain the result obtained by Shanmugam et al. [26, Theorem 5.4] and Nechita [23, Corollary 16];

(iii) Taking $n = j = 0$, $p = 1$ and $g = \frac{z}{1-z}$ in Theorem 3.1, we obtain the result obtained by Shanmugam et al. [26, Theorem 3.4] and Nechita [23, Corollary 17].

4. Superordination Results

Now, by appealing to Lemma 2.3 it is easily to prove the following theorem.

Theorem 4.1. *Let $q(z)$ be convex univalent in U with $q(0) = 1$ and $\Re\left(\frac{1}{\gamma}\right) > 0$. If $f \in \mathcal{A}(p)$ such that $\frac{\delta(p;j)z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}} \in H[q(0),1]\cap Q$,*

$$\begin{aligned}&\left[1+\gamma\left(\frac{p-j}{\lambda}\right)\right]\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}}\\&+\gamma\left(\frac{p-j}{\lambda}\right)\delta(p;j)\,z^{p-j}\left\{\frac{D_{\lambda,p}^{n+2}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}}-\frac{2\left[D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)\right]^{2}}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{3}}\right\}\end{aligned}$$

is univalent in U *and the following superordination condition*

$$q(z)+\gamma zq^{'}(z)$$
$$\prec \left[1+\gamma\left(\frac{p-j}{\lambda}\right)\right]\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}}$$
$$+\gamma\left(\frac{p-j}{\lambda}\right)\delta(p;j)\,z^{p-j}\left\{\frac{D_{\lambda,p}^{n+2}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}}-\frac{2\left[D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)\right]^{2}}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{3}}\right\}$$

holds, then

$$q(z)\prec\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}}$$

and $q(z)$ *is the best subordinant.*

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\leq B<A\leq 1)$ in Theorem 4.1, we have the following corollary.

Corollary 4.2. *Let* $\Re\left(\frac{1}{\gamma}\right)>0$ *and* $f\in\mathcal{A}(p)$ *such that* $\frac{\delta(p;j)z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}}\in H[q(0),1]\cap Q$,

$$\left[1+\gamma\left(\frac{p-j}{\lambda}\right)\right]\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}}$$
$$+\gamma\left(\frac{p-j}{\lambda}\right)\delta(p;j)\,z^{p-j}\left\{\frac{D_{\lambda,p}^{n+2}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}}-\frac{2\left[D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)\right]^{2}}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{3}}\right\}$$

is univalent in U *and the following superordination condition*

$$\frac{1+Az}{1+Bz}+\gamma\frac{(A-B)\,z}{(1+Bz)^{2}}$$
$$\prec \left[1+\gamma\left(\frac{p-j}{\lambda}\right)\right]\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}}$$
$$+\gamma\left(\frac{p-j}{\lambda}\right)\delta(p;j)\,z^{p-j}\left\{\frac{D_{\lambda,p}^{n+2}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}}-\frac{2\left[D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)\right]^{2}}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{3}}\right\}$$

holds, then

$$\frac{1+Az}{1+Bz}\prec\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^{2}}$$

and $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Taking $g=\frac{z^p}{1-z}$ in Theorem 4.1, we obtain the following corollary.

Corollary 4.3. *Let* $q(z)$ *be convex univalent in* U *with* $q(0)=1$ *and* $\Re\left(\frac{1}{\gamma}\right)>0$. *If* $f\in\mathcal{A}(p)$ *such that* $\frac{\delta(p;j)z^{p-j}D_{\lambda,p}^{n+1}f^{(j)}(z)}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^2}\in H[q(0),1]\cap Q$,

$$\left[1+\gamma\left(\frac{p-j}{\lambda}\right)\right]\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}f^{(j)}(z)}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^2}$$
$$+\gamma\left(\frac{p-j}{\lambda}\right)\delta(p;j)\,z^{p-j}\left\{\frac{D_{\lambda,p}^{n+2}f^{(j)}(z)}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^2}-\frac{2\left[D_{\lambda,p}^{n+1}f^{(j)}(z)\right]^2}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^3}\right\}$$

is univalent in U *and the following superordination condition*

$$q(z)+\gamma zq'(z)$$
$$\prec\left[1+\gamma\left(\frac{p-j}{\lambda}\right)\right]\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}f^{(j)}(z)}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^2}$$
$$+\gamma\left(\frac{p-j}{\lambda}\right)\delta(p;j)\,z^{p-j}\left\{\frac{D_{\lambda,p}^{n+2}f^{(j)}(z)}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^2}-\frac{2\left[D_{\lambda,p}^{n+1}f^{(j)}(z)\right]^2}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^3}\right\}$$

holds, then

$$q(z)\prec\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}f^{(j)}(z)}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^2}$$

and $q(z)$ *is the best subordinant.*

Remark 4.4. (i) Taking $\lambda=1$ in Corollary 4.3, we obtain the result obtained by Aouf and Seoudy [7, Theorem 2];

(ii) Taking $p=1$, $j=0$ and $g=\frac{z}{1-z}$ in Theorem 4.1, we obtain the result obtained by Shanmugam et al. [26, Theorem 5.5];

(iii) Taking $n=j=0$, $p=1$ and $g=\frac{z}{1-z}$ in Theorem 4.1, we obtain the result obtained by Shanmugam et al. [26, Theorem 3.5].

5. Sandwich Resuts

Combining Theorem 3.1 and Theorem 4.1, we get the following sandwich theorem for the linear operator $D_{\lambda,p}^{n}(f*g)^{(j)}$.

Theorem 5.1. *Let* q_1 *be convex univalent in* U *with* $q_1(0) = 1$, $\Re\left(\frac{1}{\gamma}\right) > 0$, q_2 *be univalent in* U *with* $q_2(0) = 1$ *and satisfies the inequality* (3.1). *If* $f \in \mathcal{A}(p)$ *such that* $\frac{\delta(p;j)z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^2} \in H[q(0),1] \cap Q$,

$$\left[1+\gamma\left(\frac{p-j}{\lambda}\right)\right]\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^2}$$
$$+\gamma\left(\frac{p-j}{\lambda}\right)\delta(p;j)\,z^{p-j}\left\{\frac{D_{\lambda,p}^{n+2}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^2}-\frac{2\left[D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)\right]^2}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^3}\right\}$$

is univalent in U *and*

$$q_1(z)+\gamma z q_1'(z)$$
$$\prec\left[1+\gamma\left(\frac{p-j}{\lambda}\right)\right]\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^2}$$
$$+\gamma\left(\frac{p-j}{\lambda}\right)\delta(p;j)\,z^{p-j}\left\{\frac{D_{\lambda,p}^{n+2}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^2}-\frac{2\left[D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)\right]^2}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^3}\right\}$$
$$\prec q_2(z)+\gamma z q_2'(z)$$

holds, then

$$q_1(z)\prec\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^2}\prec q_2(z),$$

q_1 *and* q_2 *are, respectively, the best subordinant and the best dominant.*

Taking $q_i(z) = \frac{1+A_iz}{1+B_iz}$ $(i = 1, 2; -1 \le B_2 \le B_1 < A_1 \le A_2 \le 1)$ in Theorem 5.1, we have the following corollary.

Corollary 5.2. *Let* $\Re\left(\frac{1}{\gamma}\right) > 0$ *and* $f \in \mathcal{A}(p)$ *such that* $\frac{\delta(p;j)z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^2} \in H[q(0),1] \cap Q$,

$$\left[1+\gamma\left(\frac{p-j}{\lambda}\right)\right]\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^2}$$
$$+\gamma\left(\frac{p-j}{\lambda}\right)\delta(p;j)\,z^{p-j}\left\{\frac{D_{\lambda,p}^{n+2}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^2}-\frac{2\left[D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)\right]^2}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^3}\right\}$$

is univalent in U and

$$\begin{aligned}
&\frac{1+A_1z}{1+B_1z}+\gamma\frac{(A_1-B_1)\,z}{(1+B_1z)^2}\\
\prec\ &\left[1+\gamma\left(\frac{p-j}{\lambda}\right)\right]\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^2}\\
&+\gamma\left(\frac{p-j}{\lambda}\right)\delta(p;j)\,z^{p-j}\left\{\frac{D_{\lambda,p}^{n+2}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^2}-\frac{2\left[D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)\right]^2}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^3}\right\}\\
\prec\ &\frac{1+A_2z}{1+B_2z}+\gamma\frac{(A_2-B_2)\,z}{(1+B_2z)^2}
\end{aligned}$$

holds, then

$$\frac{1+A_1z}{1+B_1z}\prec\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}(f*g)^{(j)}(z)}{\left[D_{\lambda,p}^{n}(f*g)^{(j)}(z)\right]^2}\prec\frac{1+A_2z}{1+B_2z},$$

$\frac{1+A_1z}{1+B_1z}$ *and* $\frac{1+A_2z}{1+B_2z}$ *are, respectively, the best subordinant and the best dominant.*

Taking $g=\frac{z^p}{1-z}$ in Theorem 5.1, we obtain the following corollary.

Corollary 5.3. *Let q_1 be convex univalent in U with $q_1(0)=1$, $\Re\left(\frac{1}{\gamma}\right)>0$, q_2 be univalent in U with $q_2(0)=1$ and satisfies the inequality (3.1). If $f\in\mathcal{A}(p)$ such that* $\frac{\delta(p;j)z^{p-j}D_{\lambda,p}^{n+1}f^{(j)}(z)}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^2}\in H[q(0),1]\cap Q$,

$$\begin{aligned}
&\left[1+\gamma\left(\frac{p-j}{\lambda}\right)\right]\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}f^{(j)}(z)}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^2}\\
&+\gamma\left(\frac{p-j}{\lambda}\right)\delta(p;j)\,z^{p-j}\left\{\frac{D_{\lambda,p}^{n+2}f^{(j)}(z)}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^2}-\frac{2\left[D_{\lambda,p}^{n+1}f^{(j)}(z)\right]^2}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^3}\right\}
\end{aligned}$$

is univalent in U and

$$\begin{aligned}
&q_1(z)+\gamma zq_1'(z)\\
\prec\ &\left[1+\gamma\left(\frac{p-j}{\lambda}\right)\right]\frac{\delta(p;j)\,z^{p-j}D_{\lambda,p}^{n+1}f^{(j)}(z)}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^2}\\
&+\gamma\left(\frac{p-j}{\lambda}\right)\delta(p;j)\,z^{p-j}\left\{\frac{D_{\lambda,p}^{n+2}f^{(j)}(z)}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^2}-\frac{2\left[D_{\lambda,p}^{n+1}f^{(j)}(z)\right]^2}{\left[D_{\lambda,p}^{n}f^{(j)}(z)\right]^3}\right\}\\
\prec\ &q_2(z)+\gamma zq_2'(z)
\end{aligned}$$

holds, then

$$q_1(z) \prec \frac{\delta(p;j)\, z^{p-j} D_{\lambda,p}^{n+1} f^{(j)}(z)}{\left[D_{\lambda,p}^{n} f^{(j)}(z)\right]^2} \prec q_2(z)\,,$$

q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Remark 5.4. (i) Taking $\lambda = 1$ in Corollary 5.3, we obtain the result obtained by Aouf and Seoudy [7, Theorem 3];

(ii) Taking $p = 1$, $j = 0$ and $g = \frac{z}{1-z}$ in Theorem 5.1, we obtain the result obtained by Shanmugam et al. [26, Theorem 5.6];

(iii) Taking $n = j = 0$, $p = 1$ and $g = \frac{z}{1-z}$ in Theorem 5.1, we obtain the result obtained by Shanmugam et al. [26, Corollary 3.6].

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