Differential Subordination and Superordination Results for Higher-Order Derivatives of $p$-Valent Functions Involving a Generalized Differential Operator

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Abstract. The purpose of this paper is to obtain some subordination, superordination and sandwich results for higher-order derivatives of $p$-valent functions involving a generalized differential operator. Some of our results generalize previously known results.

Keywords: Analytic function; Hadamard product; Differential subordination; Superordination; Sandwich theorems; Linear operator.

1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, p]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \ldots (a \in \mathbb{C}; p \in \mathbb{N} = \{1, 2, \ldots\}).$$

For simplicity $H[a] = H[a, 1]$. Also, let $A(p)$ be the subclass of $H(U)$ consisting
of functions of the form:

\[ f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}), \tag{1.1} \]

which are \( p \)-valent in \( U \). We write \( A(1) = \mathcal{A} \).

If \( f, g \in H(U) \), we say that \( f \) is subordinate to \( g \) or \( g \) is superordinate to \( f \), written \( f(z) \prec g(z) \) if there exists a Schwarz function \( w \), which (by definition) is analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) for all \( z \in U \), such that \( f(z) = g(w(z)), \, z \in U \). Furthermore, if the function \( g \) is univalent in \( U \), then we have the following equivalence, (cf., e.g. [14], [21] and [22]):

\[ f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U). \]

Let \( \phi : \mathbb{C}^2 \times U \to \mathbb{C} \) and \( h \) be univalent in \( U \). If \( \beta \) is analytic in \( U \) and satisfies the first order differential subordination:

\[ \phi \left( \beta(z), z\beta'(z); z \right) \prec h(z), \tag{1.2} \]

then \( \beta \) is a solution of the differential subordination (1.2). The univalent function \( q \) is called a dominant of the solutions of the differential subordination (1.2) if \( \beta(z) \prec q(z) \) for all \( \beta \) satisfying (1.2). A univalent dominant \( \tilde{q} \) that satisfies \( \tilde{q} \prec q \) for all dominants of (1.2) is called the best dominant. If \( \beta \) and \( \phi \) are univalent functions in \( U \) and satisfies first order differential superordination:

\[ h(z) \prec \phi \left( \beta(z), z\beta'(z); z \right), \tag{1.3} \]

then \( \beta \) is a solution of the differential superordination (1.3). An analytic function \( q \) is called a subordinant of the solutions of the differential superordination (1.3) if \( q(z) \prec \beta(z) \) for all \( \beta \) satisfying (1.3). A univalent subordinant \( \tilde{q} \) that satisfies \( q(z) \prec \tilde{q}(z) \) for all subordinates of (1.3) is called the best subordinant.

Using the results of Miller and Mocanu [22], Bulboaca [13] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [14]. Ali et al. [1], have used the results of Bulboaca [13] to obtain sufficient conditions for normalized analytic functions \( f \in \mathcal{A} \) to satisfy:

\[ q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z), \]

where \( q_1 \) and \( q_2 \) are given univalent functions in \( U \) with \( q_1(0) = q_2(0) = 1 \). Also, Tuneski [27] obtained a sufficient condition for starlikeness of \( f \in \mathcal{A} \) in terms of the quantity \( \frac{f''(z)f(z)}{(f'(z))^2} \). Recently, Shanmugam et al. [26] obtained sufficient conditions for the normalized analytic function \( f \in \mathcal{A} \) to satisfy

\[ q_1(z) \prec \frac{f(z)}{zF(z)} \prec q_2(z) \]
\[ q_1(z) \prec \frac{z^2f'(z)}{|f(z)|^2} \prec q_2(z). \]
For functions \( f \in \mathcal{A}(p) \) given by (1.1) and \( g \in \mathcal{A}(p) \) given by
\[
g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \quad (p \in \mathbb{N}),
\]
the Hadamard product (or convolution) of \( f \) and \( g \) is given by
\[
(f \ast g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g \ast f)(z).
\]

Upon differentiating both sides of (1.5) \( j \)-times with respect to \( z \), we have
\[
(f \ast g)^{(j)}(z) = \delta(p; j) z^{p-j} + \sum_{k=p+1}^{\infty} \delta(k; j) a_k b_k z^{k-j},
\]
where
\[
\delta(p; j) = \frac{p^j}{(p-j)!} \quad (p > j; p \in \mathbb{N}; j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).
\]

For functions \( f, g \in \mathcal{A}(p) \), we define the linear operator \( D_{\lambda,p}^n (f \ast g)^{(j)} : \mathcal{A}(p) \to \mathcal{A}(p) \) by:
\[
D_{\lambda,p}^n (f \ast g)^{(j)}(z) = (f \ast g)^{(j)}(z),
\]
\[
D_{\lambda,p}^1 (f \ast g)^{(j)}(z) = D_{\lambda,p} (f \ast g)^{(j)}(z)
\]
\[= (1 - \lambda) (f \ast g)^{(j)}(z) + \frac{\lambda}{p-j} z \left( (f \ast g)^{(j)} \right)'(z)
\]
\[= \delta(p; j) z^{p-j} + \sum_{k=p+1}^{\infty} \left( \frac{p-j + \lambda (k-p)}{p-j} \right) \delta(k; j) a_k b_k z^{k-j},
\]
\[
D_{\lambda,p}^2 (f \ast g)^{(j)}(z) = D \left( D_{\lambda,p}^1 (f \ast g)^{(j)} \right)(z)
\]
\[= \delta(p; j) z^{p-j} + \sum_{k=p+1}^{\infty} \left( \frac{p-j + \lambda (k-p)}{p-j} \right)^2 \delta(k; j) a_k b_k z^{k-j},
\]
and (in general)
\[
D_{\lambda,p}^n (f \ast g)^{(j)}(z) = D(D_{\lambda,p}^{n-1} (f \ast g)^{(j)}(z))
\]
\[= \delta(p; j) z^{p-j} + \sum_{k=p+1}^{\infty} \left( \frac{p-j + \lambda (k-p)}{p-j} \right)^n \delta(k; j) a_k b_k z^{k-j},
\]
\[\quad (\lambda > 0; p > j; p \in \mathbb{N}; j, n \in \mathbb{N}_0; z \in U). \tag{1.8}
\]

From (1.8), we can easily deduce that
\[
\frac{\lambda z}{p-j} \left( D_{\lambda,p}^n (f \ast g)^{(j)}(z) \right)' = D_{\lambda,p}^{n+1} (f \ast g)^{(j)}(z) - (1 - \lambda) D_{\lambda,p}^n (f \ast g)^{(j)}(z)
\]
\[\quad (\lambda > 0; p > j; p \in \mathbb{N}; j, n \in \mathbb{N}_0; z \in U). \tag{1.9}
\]
We observe that the linear operator \( D_{\lambda,p}^n (f \ast g)^{(j)}(z) \) reduces to several interesting many other linear operators considered earlier for different choices of \( j, n, \lambda \) and the function \( g \):

(i) For \( j = 0, D_{\lambda,p}^n (f \ast g)^{(0)}(z) = D_{\lambda,p}^n (f \ast g)(z) \), where the operator \( D_{\lambda,p}^n (f \ast g)(\lambda \geq 0, p \in \mathbb{N}, n \in \mathbb{N}_0) \) was introduced and studied by Selvaraj et al. [25] (see also [12]) and \( D_{\lambda,p}^n (f \ast g)(z) = D_{\lambda,p}^n (f \ast g)(z) \), where the operator \( D_{\lambda,p}^n (f \ast g) \) was introduced by Aouf [3,4] (see also [7]) and \( D_{\lambda,p}^n (f \ast g)^{(0)}(z) = D_{\lambda,p}^n (f \ast g)(z) \), where the operator \( D_{\lambda,p}^n (f \ast g) \) is the \( p \)-valent Al-Oboudi operator which was introduced by El-Ashwah and Aouf [17], \( D_{\lambda,p}^n (f \ast g)^{(j)}(z) = D_{\lambda,p}^n (f \ast g)(z) \), where the operator \( D_{\lambda,p}^n (f \ast g) \) (\( p > j, p \in \mathbb{N}, n \in \mathbb{N}_0 \)) was introduced and studied by Aouf [3,4] (see also [7]) and \( D_{\lambda,p}^n (f \ast g)^{(0)}(z) = D_{\lambda,p}^n (f \ast g)(z) \), where the operator \( D_{\lambda,p}^n (f \ast g) \) is the \( p \)-valent Sălăgean operator which was introduced and studied by Kamali and Orhan [18] (see also [5], [10] and [11]);

(ii) For

\[
g(z) = \frac{z^p}{1 - z} \quad (p \in \mathbb{N}; z \in U)
\]

we have \( D_{\lambda,p}^n (f \ast g)^{(j)}(z) = D_{\lambda,p}^n (f \ast g)(z), D_{\lambda,p}^n (f \ast g)^{(0)}(z) = D_{\lambda,p}^n (f \ast g)(z) \), where the operator \( D_{\lambda,p}^n \) is the \( p \)-valent \( \alpha \)-Oboudi operator which was introduced and studied by Dziok and Srivastava [16] and contains in turn many interesting operators;

(iii) For

\[
g(z) = z^p + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \ldots (\alpha_q)_{k-p} \cdot \beta_1 k-p \ldots \beta_s k-p}{k!} \frac{z^k}{1 - z}\]

(\( z \in U \)),

(\( \nu_k = \frac{\Gamma(\nu + k)}{\Gamma(\nu)} \) \( k = 0 \),

\( (\nu)_k = \begin{cases} 1, & \text{if } k = 0, \\ \nu(\nu + 1)(\nu + 2) \ldots (\nu + k - 1), & \text{if } k \in \mathbb{N}, \end{cases} \)

we have \( D_{\lambda,p}^n (f \ast g)^{(j)}(z) = D_{\lambda,p}^n (H_{p,q,s}(\alpha_1) f)^{(j)}(z) \) and \( D_{\lambda,p}^n (f \ast g)^{(0)}(z) = H_{p,q,s}(\alpha_1) f(z) \), where the operator \( H_{p,q,s}(\alpha_1) = H_{p,q,s}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) \) is the Dziok-Srivastava operator which was introduced and studied by Dziok and Srivastava [16] and contains in turn many interesting operators;

(iv) For

\[
g(z) = z^p + \sum_{k=p+1}^{\infty} \frac{(p + l + \alpha (k - p))_m}{p + l} \frac{z^k}{1 - z}\]

(\( \alpha \geq 0; l \geq 0; p \in \mathbb{N}; m \in \mathbb{N}_0; z \in U \)),

we have \( D_{\lambda,p}^n (f \ast g)^{(j)}(z) = D_{\lambda,p}^n (I_p(m, \alpha, l) f)^{(j)}(z) \), and \( D_{\lambda,p}^n (f \ast g)^{(0)}(z) = I_p(m, \alpha, l) f(z) \), where the operator \( I_p(m, \alpha, l) \) was introduced and studied by Cătăs [15] which contains in turn many interesting operators such as, \( I_p(m, 1, l) = I_p(m, l) \), where the operator \( I_p(m, l) \) was investigated by Kumar et al. [19];
(v) For
\[ g(z) = z^p + \frac{\Gamma(p + \alpha + \beta)}{\Gamma(p + \beta)} \sum_{k=p+1}^{\infty} \frac{\Gamma(k + \beta)}{\Gamma(k + \alpha + \beta)} z^k \] (1.13)
we have \( D_{n,\lambda,p} (f * g)^{(j)} (z) = D_{n,\lambda,p} (Q_{\alpha,\beta,p} f)^{(j)} (z) \) and \( D_{0,\lambda,p} (f * g)^{(0)} (z) = Q_{\alpha,\beta,p} f(z) \), where the operator \( Q_{\alpha,\beta,p} \) was introduced and studied by Liu and Owa [20] (see also [9]).

(vi) For \( j = 0 \) and \( g \) of the form (1.11) with \( p = 1 \), we have \( D_{n,\lambda,1} (f * g) (z) = D_{n}^n (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) (z) \), where the operator \( D_{n}^n (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) \) was introduced and studied by Selvaraj and Karthikeyan [24];

(vii) For \( j = 0 \), \( p = 1 \) and \( g(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k + 1) \Gamma(2 - m)}{\Gamma(k + 1 - m)} z^k , \) (1.14)
where \( n \in \mathbb{N}_0; 0 \leq m < 1; z \in U \), we have \( D_{n,\lambda,1} (f * g) (z) = D_{n,m}^n f(z) \), where the operator \( D_{n,m}^n \) was introduced and studied by Al-Oboudi and Al-Amoudi [2].

In this paper, we will derive several subordination, superordination and sandwich results involving the operator \( D_{n,\lambda,p} (f * g)^{(j)} \).

2. Definitions and Preliminaries

In order to prove our results, we need the following definition and lemmas.

**Definition 2.1.** [22] Denote by \( Q \), the set of all functions \( f \) that are analytic and injective on \( U \setminus E(f) \), where \( E(f) = \{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \} \), and are such that \( f'(\zeta) \neq 0 \) for \( \zeta \in \partial U \setminus E(f) \).

**Lemma 2.2.** [26] Let \( q \) be univalent function in \( U \) with \( q(0) = 1 \). Let \( \gamma_i \in \mathbb{C} (i = 1, 2) \), \( \gamma_2 \neq 0 \), further assume that
\[ \Re \left\{ 1 + \frac{z q'' (z)}{q' (z)} \right\} > \max \left\{ 0, -\Re \left( \frac{\gamma_1}{\gamma_2} \right) \right\} . \] (2.1)

If \( \beta \) is analytic function in \( U \), and
\[ \gamma_1 \beta (z) + \gamma_2 z \beta' (z) < \gamma_1 q (z) + \gamma_2 z q' (z) , \]
then \( \beta \prec q \) and \( q \) is the best dominant.
Lemma 2.3. [26] Let \( q \) be convex univalent function in \( U \), \( q(0) = 1 \). Let \( \gamma_1, \gamma_2 \in \mathbb{C}(i = 1, 2) \), \( \gamma_2 \neq 0 \) and \( \Re \left( \frac{\gamma_2}{\gamma_1} \right) > 0 \). If \( \beta \in H[q(0), 1] \cap Q \), \( \gamma_1 \beta(z) + \gamma_2 z \beta'(z) \) is univalent in \( U \) and

\[
\gamma_1 q(z) + \gamma_2 z q'(z) < \gamma_1 \beta(z) + \gamma_2 z \beta'(z),
\]

then \( q < \beta \) and \( q \) is the best subordinant.

3. Subordination Results

Unless otherwise mentioned, we assume throughout this paper that \( \gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \), \( \lambda > 0 \), \( p > j \), \( p \in \mathbb{N} \), \( n, j \in \mathbb{N}_0 \) and \( \delta(p; j) \) is given by (1.7).

Theorem 3.1. Let \( q(z) \) be univalent in \( U \) with \( q(0) = 1 \). Further, assume that

\[
\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, -\Re \left( \frac{1}{\gamma} \right) \right\},
\]

If \( f \in A(p) \) satisfies the following subordination condition:

\[
\left[ 1 + \gamma \left( \frac{p - j}{\lambda} \right) \right] \frac{\delta(p; j) z^{p-j} D_{\lambda,p}^{n+1} (f \ast g)^{(j)}(z)}{\left[ D_{\lambda,p}^{n} (f \ast g)^{(j)}(z) \right]^2} + \gamma \left( \frac{p - j}{\lambda} \right) \delta(p; j) z^{p-j} \left\{ \frac{D_{\lambda,p}^{n+2} (f \ast g)^{(j)}(z)}{\left[ D_{\lambda,p}^{n} (f \ast g)^{(j)}(z) \right]^2} - 2 \frac{D_{\lambda,p}^{n+1} (f \ast g)^{(j)}(z)}{D_{\lambda,p}^{n} (f \ast g)^{(j)}(z)} \right\} \prec q(z) + \gamma z q'(z),
\]

then

\[
\frac{\delta(p; j) z^{p-j} D_{\lambda,p}^{n+1} (f \ast g)^{(j)}(z)}{\left[ D_{\lambda,p}^{n} (f \ast g)^{(j)}(z) \right]^2} \prec q(z)
\]

and \( q(z) \) is the best dominant.

Proof. Define a function \( q(z) \) by

\[
q(z) = \frac{\delta(p; j) z^{p-j} D_{\lambda,p}^{n+1} (f \ast g)^{(j)}(z)}{\left[ D_{\lambda,p}^{n} (f \ast g)^{(j)}(z) \right]^2} \quad (z \in U).
\]

Then the function \( q \) is analytic in \( U \) and \( q(0) = 1 \). Therefore, differentiating (3.3) logarithmically with respect to \( z \) and using the identity (1.9) in the resulting
equation, we have
\[
\left[1 + \gamma \left( \frac{p - j}{\lambda} \right) \right] \delta \left( p; j \right) z^{p-j} D^{n+1}_{\lambda,p} \left( f \ast g \right)^{(j)}(z) \\
+ \gamma \left( \frac{p - j}{\lambda} \right) \delta \left( p; j \right) z^{p-j} \left\{ \frac{D^{n+2}_{\lambda,p} \left( f \ast g \right)^{(j)}(z)}{D^n_{\lambda,p} \left( f \ast g \right)^{(j)}(z)} \right\}^2 - 2 \left\{ \frac{D^{n+1}_{\lambda,p} \left( f \ast g \right)^{(j)}(z)}{D^n_{\lambda,p} \left( f \ast g \right)^{(j)}(z)} \right\}^3 \\
= q(z) + \gamma z q'(z),
\]
that is, \( q(z) + \gamma z q'(z) \prec q(z) + \gamma z q'(z) \). Therefore, Theorem 3.1 now follows by applying Lemma 2.2.

Putting \( q(z) = \frac{Az}{1+Bz} \) in Theorem 3.1, it is easy to check that the assumption (3.1) holds whenever \(-1 \leq B < A \leq 1\), hence we obtain the following corollary.

**Corollary 3.2.** Let \(-1 \leq B < A \leq 1\) and assume that
\[
\Re \left\{ \frac{1-Bz}{1+Bz} \right\} > \max \left\{ 0, -\Re \left( \frac{1}{\gamma} \right) \right\}.
\]
If \( f \in \mathcal{A}(p) \) satisfy the following subordination condition
\[
\left[1 + \gamma \left( \frac{p - j}{\lambda} \right) \right] \delta \left( p; j \right) z^{p-j} D^{n+1}_{\lambda,p} \left( f \ast g \right)^{(j)}(z) \\
+ \gamma \left( \frac{p - j}{\lambda} \right) \delta \left( p; j \right) z^{p-j} \left\{ \frac{D^{n+2}_{\lambda,p} \left( f \ast g \right)^{(j)}(z)}{D^n_{\lambda,p} \left( f \ast g \right)^{(j)}(z)} \right\}^2 - 2 \left\{ \frac{D^{n+1}_{\lambda,p} \left( f \ast g \right)^{(j)}(z)}{D^n_{\lambda,p} \left( f \ast g \right)^{(j)}(z)} \right\}^3 \\
\prec \frac{1+Az}{1+Bz} + \gamma \frac{(A-B)z}{(1+Bz)^2},
\]
then
\[
\delta \left( p; j \right) z^{p-j} D^{n+1}_{\lambda,p} \left( f \ast g \right)^{(j)}(z) \\
\left\{ \frac{D^n_{\lambda,p} \left( f \ast g \right)^{(j)}(z)}{1+Bz} \right\}^2 \prec \frac{1+Az}{1+Bz}
\]
and the function \( \frac{1+Az}{1+Bz} \) is the best dominan.

Taking \( g = \frac{z^p}{1-z} \) in Theorem 3.1, we obtain the following corollary.

**Corollary 3.3.** Let \( q \) be univalent in \( U \) with \( q(0) = 1 \) and assume that (3.1) holds.
If \( f \in A(p) \) satisfies the following subordination condition:

\[
\left[ 1 + \gamma \left( \frac{p - j}{\lambda} \right) \right] \frac{\delta (p; j) z^{p-j} D_{\lambda,p}^{n+1} f^{(j)}(z)}{\left[ D_{\lambda,p}^{n} f^{(j)}(z) \right]^2} \\
+ \gamma \left( \frac{p - j}{\lambda} \right) \delta (p; j) z^{p-j} \left\{ \frac{D_{\lambda,p}^{n+2} f^{(j)}(z)}{\left[ D_{\lambda,p}^{n} f^{(j)}(z) \right]^2} - \frac{2 D_{\lambda,p}^{n+1} f^{(j)}(z)^2}{\left[ D_{\lambda,p}^{n} f^{(j)}(z) \right]^3} \right\}
\]

\( \prec q(z) + \gamma z q'(z) \),

then

\[
\frac{\delta (p; j) z^{p-j} D_{\lambda,p}^{n+1} f^{(j)}(z)}{\left[ D_{\lambda,p}^{n} f^{(j)}(z) \right]^2} \prec q(z)
\]

and \( q(z) \) is the best dominant.

Remark 3.4. (i) Taking \( \lambda = 1 \) in Corollary 3.3, we obtain the result obtained by Aouf and Seoudy [7, Theorem 1];

(ii) Taking \( p = 1, j = 0 \) and \( g = \frac{z}{1-z} \) in Theorem 3.1, we obtain the result obtained by Shanmugam et al. [26, Theorem 5.4] and Nechita [23, Corollary 16];

(iii) Taking \( n = j = 0, p = 1 \) and \( g = \frac{z}{1-z} \) in Theorem 3.1, we obtain the result obtained by Shanmugam et al. [26, Theorem 3.4] and Nechita [23, Corollary 17].

4. Superordination Results

Now, by appealing to Lemma 2.3 it is easily to prove the following theorem.

**Theorem 4.1.** Let \( q(z) \) be convex univalent in \( U \) with \( q(0) = 1 \) and \( \Re \left( \frac{1}{z} \right) > 0 \). If \( f \in A(p) \) such that

\[
\left[ 1 + \gamma \left( \frac{p - j}{\lambda} \right) \right] \frac{\delta (p; j) z^{p-j} D_{\lambda,p}^{n+1} (f \ast g)^{(j)}(z)}{\left[ D_{\lambda,p}^{n} (f \ast g)^{(j)}(z) \right]^2} \\
+ \gamma \left( \frac{p - j}{\lambda} \right) \delta (p; j) z^{p-j} \left\{ \frac{D_{\lambda,p}^{n+2} (f \ast g)^{(j)}(z)}{\left[ D_{\lambda,p}^{n} (f \ast g)^{(j)}(z) \right]^2} - \frac{2 D_{\lambda,p}^{n+1} (f \ast g)^{(j)}(z)^2}{\left[ D_{\lambda,p}^{n} (f \ast g)^{(j)}(z) \right]^3} \right\}
\]

\( \in H \left[ q(0) \right. , 1 \right. \cap Q, \)
is univalent in $U$ and the following superordination condition

$$q(z) + \gamma z q'(z)$$

$$\prec \left[ 1 + \gamma \left( -\frac{j}{\lambda} \right) \right] \frac{\delta (p; j) z^{p-j} D_{\lambda,p}^{n+1} (f * g)^{(j)} (z)}{[D_{\lambda,p}^{n} (f * g)^{(j)} (z)]^2}$$

$$+ \gamma \left( -\frac{j}{\lambda} \right) \delta (p; j) z^{p-j} \frac{D_{\lambda,p}^{n+2} (f * g)^{(j)} (z)}{[D_{\lambda,p}^{n} (f * g)^{(j)} (z)]^2} - 2 \frac{D_{\lambda,p}^{n+1} (f * g)^{(j)} (z)}{[D_{\lambda,p}^{n} (f * g)^{(j)} (z)]^3}$$

holds, then

$$q(z) < \frac{\delta (p; j) z^{p-j} D_{\lambda,p}^{n+1} (f * g)^{(j)} (z)}{[D_{\lambda,p}^{n} (f * g)^{(j)} (z)]^2}$$

and $q(z)$ is the best subordinant.

Taking $q(z) = \frac{1 + \frac{A}{1 + Bz}}{1 + Bz}$ ($-1 < B < A < 1$) in Theorem 4.1, we have the following corollary.

**Corollary 4.2.** Let $\Re \left( \frac{A}{B} \right) > 0$ and $f \in A(p)$ such that $\frac{\delta (p; j) z^{p-j} D_{\lambda,p}^{n+1} (f * g)^{(j)} (z)}{[D_{\lambda,p}^{n} (f * g)^{(j)} (z)]^2} \in H [q(0), 1] \cap Q$,

$$\left[ 1 + \gamma \left( -\frac{j}{\lambda} \right) \right] \frac{\delta (p; j) z^{p-j} D_{\lambda,p}^{n+1} (f * g)^{(j)} (z)}{[D_{\lambda,p}^{n} (f * g)^{(j)} (z)]^2}$$

$$+ \gamma \left( -\frac{j}{\lambda} \right) \delta (p; j) z^{p-j} \frac{D_{\lambda,p}^{n+2} (f * g)^{(j)} (z)}{[D_{\lambda,p}^{n} (f * g)^{(j)} (z)]^2} - 2 \frac{D_{\lambda,p}^{n+1} (f * g)^{(j)} (z)}{[D_{\lambda,p}^{n} (f * g)^{(j)} (z)]^3}$$

is univalent in $U$ and the following superordination condition

$$\frac{1 + A z}{1 + B z} + \gamma \frac{(A - B) z}{(1 + B z)^2}$$

$$\prec \left[ 1 + \gamma \left( -\frac{j}{\lambda} \right) \right] \frac{\delta (p; j) z^{p-j} D_{\lambda,p}^{n+1} (f * g)^{(j)} (z)}{[D_{\lambda,p}^{n} (f * g)^{(j)} (z)]^2}$$

$$+ \gamma \left( -\frac{j}{\lambda} \right) \delta (p; j) z^{p-j} \frac{D_{\lambda,p}^{n+2} (f * g)^{(j)} (z)}{[D_{\lambda,p}^{n} (f * g)^{(j)} (z)]^2} - 2 \frac{D_{\lambda,p}^{n+1} (f * g)^{(j)} (z)}{[D_{\lambda,p}^{n} (f * g)^{(j)} (z)]^3}$$

holds, then

$$\frac{1 + A z}{1 + B z} < \frac{\delta (p; j) z^{p-j} D_{\lambda,p}^{n+1} (f * g)^{(j)} (z)}{[D_{\lambda,p}^{n} (f * g)^{(j)} (z)]^2}$$
and $\frac{1+A2}{1-A2}$ is the best subordinant.

Taking $g = \frac{z}{1-z}$ in Theorem 4.1, we obtain the following corollary.

**Corollary 4.3.** Let $q(z)$ be convex univalent in $U$ with $q(0) = 1$ and $\Re\left(\frac{1}{z}\right) > 0$. If $f \in A(p)$ such that

$$
\left[1 + \gamma \left(\frac{p-j}{\lambda}\right)\right] \frac{\delta(p,j) z^{p-j} D_{\Lambda,p}^{n+1} f^{(j)}(z)}{D_{\Lambda,p}^{n} f^{(j)}(z)} + \gamma \left(\frac{p-j}{\lambda}\right) \delta(p,j) z^{p-j} \left\{ \frac{D_{\Lambda,p}^{n+2} f^{(j)}(z)}{D_{\Lambda,p}^{n} f^{(j)}(z)} - 2 \frac{D_{\Lambda,p}^{n+1} f^{(j)}(z)}{D_{\Lambda,p}^{n} f^{(j)}(z)} \right\}^2
$$

is univalent in $U$ and the following superordination condition

$$
q(z) + \gamma z q'(z) < \left[1 + \gamma \left(\frac{p-j}{\lambda}\right)\right] \frac{\delta(p,j) z^{p-j} D_{\Lambda,p}^{n+1} f^{(j)}(z)}{D_{\Lambda,p}^{n} f^{(j)}(z)} + \gamma \left(\frac{p-j}{\lambda}\right) \delta(p,j) z^{p-j} \left\{ \frac{D_{\Lambda,p}^{n+2} f^{(j)}(z)}{D_{\Lambda,p}^{n} f^{(j)}(z)} - 2 \frac{D_{\Lambda,p}^{n+1} f^{(j)}(z)}{D_{\Lambda,p}^{n} f^{(j)}(z)} \right\}^2
$$

holds, then

$$
q(z) < \frac{\delta(p,j) z^{p-j} D_{\Lambda,p}^{n+1} f^{(j)}(z)}{D_{\Lambda,p}^{n} f^{(j)}(z)}
$$

and $q(z)$ is the best subordinant.

**Remark 4.4.** (i) Taking $\lambda = 1$ in Corollary 4.3, we obtain the result obtained by Aouf and Seoudy [7, Theorem 2];

(ii) Taking $p = 1$, $j = 0$ and $g = \frac{z}{1-z}$ in Theorem 4.1, we obtain the result obtained by Shanmugam et al. [26, Theorem 5.5];

(iii) Taking $n = j = 0$, $p = 1$ and $g = \frac{z}{1-z}$ in Theorem 4.1, we obtain the result obtained by Shanmugam et al. [26, Theorem 3.5].

5. Sandwich Results

Combining Theorem 3.1 and Theorem 4.1, we get the following sandwich theorem for the linear operator $D_{\Lambda,p}^{n} (f * g)^{(j)}$. 

Theorem 5.1. Let $q_1$ be convex univalent in $U$ with $q_1(0) = 1$, $\Re\left(\frac{1}{\lambda}\right) > 0$, $q_2$ be univalent in $U$ with $q_2(0) = 1$ and satisfies the inequality (3.1). If $f \in A(p)$ such that
\[
\frac{\delta(p,j)z^{p-j}D^{n+1}_{\lambda,p}(f \ast g)^{(j)}(z)}{[D^{n}_{\lambda,p}(f \ast g)^{(j)}(z)]^2} \in H [q(0), 1] \cap Q,
\]

\[
1 + \gamma \left(\frac{p-j}{\lambda}\right) \delta(p,j)z^{p-j}D^{n+1}_{\lambda,p}(f \ast g)^{(j)}(z) + \gamma \left(\frac{p-j}{\lambda}\right) \delta(p,j)z^{p-j}\left\{\left[\frac{D^{n+2}_{\lambda,p}(f \ast g)^{(j)}(z)}{D^{n}_{\lambda,p}(f \ast g)^{(j)}(z)}\right]^2 - 2 \left[\frac{D^{n+1}_{\lambda,p}(f \ast g)^{(j)}(z)}{D^{n}_{\lambda,p}(f \ast g)^{(j)}(z)}\right]^2\right\}
\]
is univalent in $U$ and
\[
q_1(z) + \gamma q_1'(z) \prec 1 + \gamma \left(\frac{p-j}{\lambda}\right) \delta(p,j)z^{p-j}D^{n+1}_{\lambda,p}(f \ast g)^{(j)}(z) + \gamma \left(\frac{p-j}{\lambda}\right) \delta(p,j)z^{p-j}\left\{\left[\frac{D^{n+2}_{\lambda,p}(f \ast g)^{(j)}(z)}{D^{n}_{\lambda,p}(f \ast g)^{(j)}(z)}\right]^2 - 2 \left[\frac{D^{n+1}_{\lambda,p}(f \ast g)^{(j)}(z)}{D^{n}_{\lambda,p}(f \ast g)^{(j)}(z)}\right]^2\right\} \prec q_2(z) + \gamma q_2'(z)
\]
holds, then
\[
q_1(z) \prec \frac{\delta(p,j)z^{p-j}D^{n+1}_{\lambda,p}(f \ast g)^{(j)}(z)}{[D^{n}_{\lambda,p}(f \ast g)^{(j)}(z)]^2} \prec q_2(z),
\]
$q_1$ and $q_2$ are, respectively, the best subordinate and the best dominant.

Taking $q_i(z) = \frac{1 + A_i z^j}{1 + B_i z^j} (i = 1, 2; -1 \leq B_2 < B_1 < A_1 \leq A_2 \leq 1)$ in Theorem 5.1, we have the following corollary.

Corollary 5.2. Let $\Re\left(\frac{1}{\gamma}\right) > 0$ and $f \in A(p)$ such that
\[
\frac{\delta(p,j)z^{p-j}D^{n+1}_{\lambda,p}(f \ast g)^{(j)}(z)}{[D^{n}_{\lambda,p}(f \ast g)^{(j)}(z)]^2} \in H [q(0), 1] \cap Q,
\]

\[
1 + \gamma \left(\frac{p-j}{\lambda}\right) \delta(p,j)z^{p-j}D^{n+1}_{\lambda,p}(f \ast g)^{(j)}(z) + \gamma \left(\frac{p-j}{\lambda}\right) \delta(p,j)z^{p-j}\left\{\left[\frac{D^{n+2}_{\lambda,p}(f \ast g)^{(j)}(z)}{D^{n}_{\lambda,p}(f \ast g)^{(j)}(z)}\right]^2 - 2 \left[\frac{D^{n+1}_{\lambda,p}(f \ast g)^{(j)}(z)}{D^{n}_{\lambda,p}(f \ast g)^{(j)}(z)}\right]^2\right\}
\]
in $H [q(0), 1] \cap Q$. 


Corollary 5.3

Let $q_1$ be convex univalent in $U$ with $q_1(0) = 1$, $\Re \left( \frac{z}{z_0} \right) > 0$, $q_2$ be univalent in $U$ with $q_2(0) = 1$ and satisfies the inequality (3.1). If $f \in A(p)$ such that $\frac{\delta(p,j)z^{p-j}D^{n+1}_{\lambda,p}f^{(j)}(z)}{[D^{n}_{\lambda,p}f^{(j)}(z)]^2} \in H[q(0),1] \cap Q$,

\[
\begin{align*}
&\left[ 1 + \gamma \left( \frac{p-j}{\lambda} \right) \right] \frac{\delta(p,j)z^{p-j}D^{n+1}_{\lambda,p}f^{(j)}(z)}{[D^{n}_{\lambda,p}f^{(j)}(z)]^2} \\
&+ \gamma \left( \frac{p-j}{\lambda} \right) \delta(p,j)z^{p-j} \left\{ \frac{D^{n+2}_{\lambda,p}f^{(j)}(z)}{[D^{n}_{\lambda,p}f^{(j)}(z)]^2} - 2 \left[ \frac{D^{n+1}_{\lambda,p}f^{(j)}(z)}{[D^{n}_{\lambda,p}f^{(j)}(z)]^2} \right]^2 \right\}
\end{align*}
\]

is univalent in $U$ and

\[
\begin{align*}
&\left[ 1 + \gamma \left( \frac{p-j}{\lambda} \right) \right] \frac{\delta(p,j)z^{p-j}D^{n+1}_{\lambda,p}f^{(j)}(z)}{[D^{n}_{\lambda,p}f^{(j)}(z)]^2} \\
&+ \gamma \left( \frac{p-j}{\lambda} \right) \delta(p,j)z^{p-j} \left\{ \frac{D^{n+2}_{\lambda,p}f^{(j)}(z)}{[D^{n}_{\lambda,p}f^{(j)}(z)]^2} - 2 \left[ \frac{D^{n+1}_{\lambda,p}f^{(j)}(z)}{[D^{n}_{\lambda,p}f^{(j)}(z)]^2} \right]^2 \right\}
\end{align*}
\]

\[\times q_1(z) + \gamma z q'_1(z)\]

\[\times q_2(z) + \gamma z q'_2(z)\]
holds, then
\[ q_1(z) \prec \delta(p,j) z^{p-j} D^{n+1}_\lambda f^{(j)}(z) \prec q_2(z), \]

\( q_1 \) and \( q_2 \) are, respectively, the best subordinant and the best dominant.

Remark 5.4. (i) Taking \( \lambda = 1 \) in Corollary 5.3, we obtain the result obtained by Aouf and Seoudy [7, Theorem 3];

(ii) Taking \( p = 1, j = 0 \) and \( g = \frac{1}{1-z} \) in Theorem 5.1, we obtain the result obtained by Shanmugam et al. [26, Theorem 5.6];

(iii) Taking \( n = j = 0, p = 1 \) and \( g = \frac{1}{1-z} \) in Theorem 5.1, we obtain the result obtained by Shanmugam et al. [26, Corollary 3.6].

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References