Reprinted from

PHYSICS OF PLASMAS

Vol. 2, No. 4, April 1995

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pp. 1206-1216

AMERICAN INSTITUTE OF PHYSICS

The plasma transport equations derived by multiple time-scale expansions. II. An application

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(Received 28 October 1994; accepted 28 November 1994)

A multiple time scale derivative expansion scheme has been employed to reveal some aspects of Taylor's relaxation theory, which states that a "*slightly imperfect*" plasma relaxes under the conservation of the overall magnetic helicity K toward a state of minimum magnetic energy. The purpose of this paper is to investigate the time evolution of K on the ideal magnetohydrodynamic (IMHD), the MHD-collision (CMHD), and resistive diffusion (RDMHD) time scales. On the ideal MHD time scale, it is found, just as expected, that K is an invariant of motion for each single flux tube. On the MHD-collision time scale Taylor's conjecture is explicitly proven, namely that only the overall K is an invariant of motion. Finally for the resistive diffusion time scale, it is found that the time derivative is proportional to the resistivity, however, with additional terms arising from the MHD fluctuation spectrum. © *1995 American Institute of Physics.*

I. INTRODUCTION

Many theories have been presented to describe the plasma relaxation processes. However, the most famous one is that presented by Taylor through his work.[1,2] Taylor's theory states that, after an initial violently unstable phase the plasma relaxes, under the topological constraint of helicity conservation, $K=\int \mathbf{A}\cdot\mathbf{B}\, d\tau=$const into a quiescent, grossly stable state of minimum magnetic energy. Since for an ideal plasma, the magnetic field lines are frozen to the plasma, i.e., each field line maintains its own identity, Taylor concluded that K represents an infinity of topological constraints, in the sense that K is conserved for each flux tube. Furthermore, he conjectured that for a "slightly nonideal" plasma, in the sense that the topological properties of the field lines are no longer preserved, the global K is the only topological invariant of motion. Taylor's theory finds further support in experimental data from reversed-field pinches, e.g., from ZETA or HBTX. In spite of the success of this theory, there remains the important unanswered question of what is a "*slightly nonideal*" plasma and how and when may such a plasma preserve only the global K as an invariant of motion. In this paper, this question is answered on the basis of the multiple time scale approach presented by Edenstrasser (Part I). The invariance properties and time evolution of the helicity integral K are investigated on the ideal magnetohydrodynamic (IMHD), the MHD-collision (CMHD), and on the resistive diffusion (RDMHD) time scale.

II. THE MULTIPLE TIME SCALE EXPANSION OF THE HELICITY

For an ideal MHD plasma the magnetic helicity integral is conserved (locally) for each single flux tube and thus—if the wall is identical with the outermost magnetic surface—also (globally) over the whole plasma volume, while due to Taylor's conjecture for a *slightly nonideal* plasma it is conserved only globally. We are thus motivated first to define the local magnetic helicity integral K^{ψ} as well as the overall K, and apply then the multiple time scale derivative expansion scheme Eq. (13) from Eq. (13) in Part I, to obtain the explicit expression for the time evolution on each time scale.

A. The local magnetic helicity integral K^{ψ}

Based on our expansion scheme we draw the following definition for the local magnetic helicity integral K^{ψ}:

$$K^{\psi}=\int (\mathbf{A}_0+\delta\mathbf{A}_1+\delta^2\mathbf{A}_2)\cdot(\mathbf{B}_0+\delta\mathbf{B}_1+\delta^2\mathbf{B}_2) \times(d\tau_0+\delta\, d\tau_1+\delta^2\, d\tau_2), \tag{1}$$

where $d\tau=(d\tau_0+\delta d\tau_1+\delta^2\, d\tau_2)$ is (cf. Appendix A) a volume element enclosed between two neighbouring magnetic surfaces $\Psi=\Psi_0+\delta\Psi_1+\delta^2\Psi_2=$const, created by the magnetic field $\mathbf{B}=\mathbf{B}_0+\delta\mathbf{B}_1+\delta^2\mathbf{B}_2$. For the local magnetic helicity K^{ψ}, we have the analogous expansion $K^{\psi}=K_0^{\psi}+\delta K_1^{\psi}+\delta^2 K_2^{\psi}+\cdots$, with

$$K_n^{\psi}=\sum_{m=0}^{n}\int\left(\sum_{s=0}^{n-m}(\mathbf{A}_s\cdot\mathbf{B}_{n-s})\right)d\tau_m. \tag{2}$$

By applying the multiple time scale derivative expansion scheme, Eq. (13) of Part I, we obtain

$$\frac{\partial K^{\psi}}{\partial t}=\sum_{n=0}\delta^n\sum_{s=0}^{n}\frac{\partial K_{n-s}^{\psi}}{\partial t_s} =\frac{\partial K_0^{\psi}}{\partial t_0}+\delta\left.\frac{\partial K^{\psi}}{\partial t}\right|_{\mathrm{IMHD}} +\delta^2\left.\frac{\partial K^{\psi}}{\partial t}\right|_{\mathrm{CMHD}}+\cdots=\sum_{n=0}\delta^n(TK^{\psi})_n, \tag{3}$$

where $(TK^{\psi})_n$, referring to the nth-order dimensionless time evolution of the local K^{ψ}, can be written in the following general form:

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$$(TK^{\psi})_n=\sum_{i=0}^{n}\int \mathbf{A}_i\cdot\left[\sum_{j=0}^{n-i}\left(\sum_{s=0}^{n-i-j}\frac{\partial \mathbf{B}_{n-i-j-s}}{\partial t_s}\right)d\tau_j\right]$$
$$+\sum_{i=0}^{n}\int \mathbf{B}_i\cdot\left[\sum_{j=0}^{n-i}\left(\sum_{s=0}^{n-i-j}\frac{\partial \mathbf{A}_{n-i-j-s}}{\partial t_s}\right)d\tau_j\right]$$
$$+\sum_{i=0}^{n}\int\left(\sum_{s=0}^{i}\mathbf{A}_s\cdot\mathbf{B}_{i-s}\right)\cdot\left(\sum_{j=0}^{n-i}\frac{\partial\tau_{n-i-j}}{\partial t_j}\right). \tag{4}$$

The last term on the right-hand side of Eq. (4) arises on account of the time evolution of the magnetic surfaces.

B. The global magnetic helicity integral

If the rigid wall is identical with the outermost magnetic surface, then we obtain from the definition for the local magnetic helicity integral K^{ψ} of Eq. (1) the corresponding expression for the global magnetic helicity integral K,

$$K=\int(\mathbf{A}_0+\delta\mathbf{A}_1+\delta^2\mathbf{A}_2)\cdot(\mathbf{B}_0+\delta\mathbf{B}_1+\delta^2\mathbf{B}_2)d\tau, \tag{5}$$

where the integration is now taken over the whole volume of the plasma. For the global helicity K we have the analogous ordering $K=K_0+\delta K_1+\delta^2K_2+\cdots$, with

$$K_n=\sum_{m=0}^{n}\int(\mathbf{A}_m\cdot\mathbf{B}_{n-m})d\tau. \tag{6}$$

By applying the multiple time scale derivative expansion scheme Eq. (13) of Part I, we obtain, for the time evolution,

$$\frac{\partial K}{\partial t}=\sum_{n=0}\delta^n\sum_{s=0}^{n}\frac{\partial K_{n-s}}{\partial t_s}$$
$$=\frac{\partial K_0}{\partial t_0}+\delta\left.\frac{\partial K}{\partial t}\right|_{\text{IMHD}}+\delta^2\left.\frac{\partial K}{\partial t}\right|_{\text{CMHD}}+\delta^3\left.\frac{\partial K}{\partial t}\right|_{\text{RDMHD}}$$
$$=\sum_{n=0}\delta^n(TK)_n, \tag{7}$$

where the nth-order dimensionless time evolution $(TK)_n$ can be written in the following general form:

$$(TK)_n=\sum_{m=0}^{n}\int\mathbf{A}_m\cdot\left(\sum_{s=0}^{n-m}\frac{\partial\mathbf{B}_{n-m-s}}{\partial t_s}\right)d\tau$$
$$+\sum_{m=0}^{n}\int\mathbf{B}_m\cdot\left(\sum_{s=0}^{n-m}\frac{\partial\mathbf{A}_{n-m-s}}{\partial t_s}\right)d\tau. \tag{8}$$

In Eq. (8), it is obvious that the plasma is assumed to be surrounded by a rigid wall.

III. THE INVARIANCE OF THE HELICITY ON THE IMHD TIME SCALE

To prove Taylor's conjecture, we are thus motivated to investigate the time evolution of the magnetic helicity on the different time scales. By employing the above expansion scheme we first show, for the IMHD time scale, the invariance of both the local and the global magnetic helicity integrals.

First, we verify the known invariance of the local magnetic helicity integral K^{ψ} on the Alfvén time scale. By employing Eq. (4) for the nth-order dimensionless time evolution of the local magnetic helicity $(TK^{\psi})_n$, we obtain, for $n=1$,

$$\left.\frac{\partial K^{\psi}}{\partial t}\right|_{\text{IMHD}}=\int\mathbf{A}_0\cdot\left(\frac{\partial\mathbf{B}_0}{\partial t_1}+\frac{\partial\mathbf{B}_1}{\partial t_0}\right)d\tau_0+\int\mathbf{B}_0\cdot\left(\frac{\partial\mathbf{A}_0}{\partial t_1}\right.$$
$$\left.+\frac{\partial\mathbf{A}_1}{\partial t_0}\right)d\tau_0+\int(\mathbf{A}_0\cdot\mathbf{B}_0)\left(\frac{\partial d\tau_0}{\partial t_1}+\frac{\partial d\tau_1}{\partial t_0}\right)$$
$$+\int(\mathbf{A}_0\cdot\mathbf{B}_1+\mathbf{A}_1\cdot\mathbf{B}_0)\frac{\partial d\tau_0}{\partial t_0}. \tag{9}$$

With the help of the dimensionless Maxwell's equations Eq. (14) of Part I, together with Eqs. (C4), (C5a), and (D5) from the Appendices, we obtain

$$\left.\frac{\partial K^{\psi}}{\partial t}\right|_{\text{IMHD}}=\left(\int\mathbf{A}_0\times\mathbf{E}_0\cdot d\mathbf{s}_0-2\int\mathbf{B}_0\cdot\mathbf{E}_0\,d\tau_0\right.$$
$$\left.-\int(\mathbf{A}_0\cdot\mathbf{B}_0)\mathbf{u}_0\cdot d\mathbf{s}_0\right)(\Omega_i\tau_A\delta)^{-1}$$
$$+\int\mathbf{B}_0\cdot\nabla X_0\,d\tau_0, \tag{10}$$

where $\mathbf{E}_0=-\mathbf{u}_0\times\mathbf{B}_0$ and X_0 is the zeroth-order gauge potential function. Under the assumption that the plasma has relaxed to its zeroth-order equilibrium state, (i.e., $\mathbf{j}_0=\lambda\mathbf{B}_0$ and $\mathbf{u}_0=\mu\mathbf{B}_0$), X_0 has to be a single-valued function over the toroidal shell $d\tau_0$ (cf. Appendix D), and Eq. (10) reduces to

$$\left.\frac{\partial K^{\psi}}{\partial t}\right|_{\text{IMHD}}\equiv 0. \tag{11}$$

This means that for the IMHD time scale we have obtained Taylor's result, i.e., that K^{ψ} is conserved for each single flux tube.

Now we consider the case of the global magnetic helicity integral K, assuming that the plasma is surrounded by a rigid, perfectly conducting wall, i.e., the plasma boundary is assumed to be identical with the outermost magnetic surface. From Eq. (8) we obtain, for the time evolution,

$$\left.\frac{\partial K}{\partial t}\right|_{\text{IMHD}}=\int\mathbf{A}_0\cdot\left(\frac{\partial\mathbf{B}_0}{\partial t_1}+\frac{\partial\mathbf{B}_1}{\partial t_0}\right)d\tau+\int\mathbf{B}_0$$
$$\cdot\left(\frac{\partial\mathbf{A}_0}{\partial t_1}+\frac{\partial\mathbf{A}_1}{\partial t_0}\right)d\tau. \tag{12}$$

With the help of the dimensionless Maxwell's equations, Eq. (14) of Part I, together with Eq. (D5), Eq. (12), can be rewritten in the form

$$\left.\frac{\partial K}{\partial t}\right|_{\text{IMHD}}=\left(\oint \mathbf{A}_0\times\mathbf{E}_0\cdot d\mathbf{S}-2\int \mathbf{B}_0\cdot\mathbf{E}_0\,d\tau\right)(\Omega_i\tau_A\delta)^{-1}+\int \mathbf{B}_0\cdot\nabla X_0\,d\tau. \tag{13}$$

The application of the boundary conditions $\mathbf{B}_n\cdot d\mathbf{S}=\mathbf{j}_n\cdot d\mathbf{S}=\mathbf{u}_n\cdot d\mathbf{S}=0$ then yields

$$\left.\frac{\partial K}{\partial t}\right|_{\text{IMHD}}\equiv 0. \tag{14}$$

This means that the global magnetic helicity integral K is also conserved on the IMHD time scale. Thus, we have finally arrived at the well-known results concerning the invariance of the magnetic helicity integral on the IMHD time scale.

IV. THE INVARIANCE OF THE HELICITY ON THE CMHD TIME SCALE

First, we investigate the time evolution of the local magnetic helicity integral K^{ψ} on the MHD-collision (CMHD) time scale. The next order in our expansion scheme (4) yields, for the time derivative on the MHD-collision (CMHD) time scale, the expression

$$\left.\frac{\partial K^{\psi}}{\partial t}\right|_{\text{CMHD}}=\int\left[\mathbf{A}_0\cdot\left(\frac{\partial\mathbf{B}_0}{\partial t_2}+\frac{\partial\mathbf{B}_1}{\partial t_1}+\frac{\partial\mathbf{B}_2}{\partial t_0}\right)+\mathbf{B}_0\cdot\left(\frac{\partial\mathbf{A}_0}{\partial t_2}+\frac{\partial\mathbf{A}_1}{\partial t_1}+\frac{\partial\mathbf{A}_2}{\partial t_0}\right)\right]d\tau_0+\int\left[\mathbf{A}_0\cdot\left(\frac{\partial\mathbf{B}_0}{\partial t_1}+\frac{\partial\mathbf{B}_1}{\partial t_0}\right)+\mathbf{B}_0\cdot\left(\frac{\partial\mathbf{A}_0}{\partial t_1}+\frac{\partial\mathbf{A}_1}{\partial t_0}\right)\right]d\tau_1$$
$$+\int\left[\mathbf{A}_1\cdot\left(\frac{\partial\mathbf{B}_0}{\partial t_1}+\frac{\partial\mathbf{B}_1}{\partial t_0}\right)+\mathbf{B}_1\cdot\left(\frac{\partial\mathbf{A}_0}{\partial t_1}+\frac{\partial\mathbf{A}_1}{\partial t_0}\right)\right]d\tau_0+\int(\mathbf{A}_0\cdot\mathbf{B}_0)\left(\frac{\partial d\tau_0}{\partial t_2}+\frac{\partial d\tau_1}{\partial t_1}+\frac{\partial d\tau_2}{\partial t_0}\right)$$
$$+\int(\mathbf{A}_0\cdot\mathbf{B}_1+\mathbf{A}_1\cdot\mathbf{B}_0)\left(\frac{\partial d\tau_0}{\partial t_1}+\frac{\partial d\tau_1}{\partial t_0}\right). \tag{15}$$

With the help of the dimensionless Maxwell's equations, Eq. (14) of Part I, the equation can be written in the form

$$\left.\frac{\partial K^{\psi}}{\partial t}\right|_{\text{CMHD}}=\left(\oint \mathbf{A}_0\times\mathbf{E}_1\cdot d\mathbf{s}_0-2\int \mathbf{B}_0\cdot\mathbf{E}_1\,d\tau_0\right)(\Omega_i\tau_A\delta)^{-1}+\int \mathbf{B}_0\cdot\nabla X_1\,d\tau_0+\left(\oint \mathbf{A}_0\times\mathbf{E}_0\cdot d\mathbf{s}_1-2\int \mathbf{B}_0\cdot\mathbf{E}_0\,d\tau_1\right)$$
$$\times(\Omega_i\tau_A\delta)^{-1}+\int \mathbf{B}_0\cdot\nabla X_0\,d\tau_1+\left(\oint \mathbf{A}_1\times\mathbf{E}_0\cdot d\mathbf{s}_0-2\int \mathbf{B}_1\cdot\mathbf{E}_0\,d\tau_0\right)(\Omega_i\tau_A\delta)^{-1}+\int \mathbf{B}_1\cdot\nabla X_0\,d\tau_0$$
$$+\int(\mathbf{A}_0\cdot\mathbf{B}_0)\left(\frac{\partial d\tau_0}{\partial t_2}+\frac{\partial d\tau_1}{\partial t_1}+\frac{\partial d\tau_2}{\partial t_0}\right)+\int(\mathbf{A}_0\cdot\mathbf{B}_1+\mathbf{A}_1\cdot\mathbf{B}_0)\left(\frac{\partial d\tau_0}{\partial t_1}+\frac{\partial d\tau_1}{\partial t_0}\right). \tag{16}$$

Under the assumption that the plasma has relaxed to its zeroth-order equilibrium state (i.e., $\mathbf{j}_0=\lambda\mathbf{B}_0$ and $\mathbf{u}_0=\mu\mathbf{B}_0$), Eq. (16) reduces to

$$\left.\frac{\partial K^{\psi}}{\partial t}\right|_{\text{CMHD}}=\left(\oint \mathbf{A}_0\times\mathbf{E}_1\cdot d\mathbf{s}_0-2\int \mathbf{B}_0\cdot\mathbf{E}_1\,d\tau_0\right)\times(\Omega_i\tau_A\delta)^{-1}-\int \mathbf{B}_0\cdot\nabla X_1\,d\tau_0+\int(\mathbf{A}_0\cdot\mathbf{B}_0)\left(\frac{\partial d\tau_0}{\partial t_2}+\frac{\partial d\tau_1}{\partial t_1}+\frac{\partial d\tau_2}{\partial t_0}\right). \tag{17}$$

The application of the first-order dimensionless Ohm's law, Eq. (24) of Part I, to the second term on the right-hand side (RHS) of Eq. (17), together with the assumption that the plasma behaves along the field lines like a polytropic medium, i.e., that $\nabla p_{e0}/n_{e0}$ can be considered as the gradient of a function of n_{e0}, $\nabla p_{e0}/n_{e0}=\nabla f(n_{e0})$, then yields

$$\int \mathbf{B}_0\cdot\mathbf{E}_1\,d\tau_0=\frac{1}{a_e}\left(\frac{\delta_e}{\delta_i}\right)^{1/2}\int \mathbf{B}_0\cdot\frac{\nabla p_{e0}}{n_{e0}}\,d\tau_0=0. \tag{18a}$$

The third term on the RHS of Eq. (17), which is due to the multiple valuedness of the gauge potential function X_1 (cf. Appendix D), can be written as

$$\int \mathbf{B}_0\cdot\nabla X_1\,d\tau_0=-\oint \mathbf{A}_0\times\nabla X_1\cdot d\mathbf{s}_0=\oint \mathbf{A}_0\times\left[(\Omega_i\tau_A\delta)^{-1}\mathbf{E}_1+\left(\frac{\partial\mathbf{A}_0}{\partial t_2}+\frac{\partial\mathbf{A}_1}{\partial t_1}+\frac{\partial\mathbf{A}_2}{\partial t_0}\right)\right]\cdot d\mathbf{s}_0. \tag{18b}$$

With the help of Eqs. (C6a) and (C6b) from Appendix C, the last term on the RHS can be either written in the form

$$\int (\mathbf{A}_0\cdot\mathbf{B}_0)\left(\frac{\partial d\tau_0}{\partial t_2}+\frac{\partial d\tau_1}{\partial t_1}+\frac{\partial d\tau_2}{\partial t_0}\right)$$

$$=-\oint (\mathbf{A}_0\cdot\mathbf{B}_0)[\mathbf{u}_1\cdot d\mathbf{s}_0-\mu\mathbf{B}_1\cdot d\mathbf{s}_0$$

$$+(\Omega_i\tau_A n_0)^{-1}(\mathbf{j}_1-\lambda\mathbf{B}_1)\cdot d\mathbf{s}_0](\Omega_i\tau_A\delta)^{-1}, \tag{18c}$$

or, equivalently,

$$\int (\mathbf{A}_0\cdot\mathbf{B}_0)\left(\frac{\partial d\tau_0}{\partial t_2}+\frac{\partial d\tau_1}{\partial t_1}+\frac{\partial d\tau_2}{\partial t_0}\right)$$

$$=-\oint (\mathbf{A}_0\cdot\mathbf{B}_0)\{\mathbf{u}_1\cdot d\mathbf{s}_0-\mu\mathbf{B}_1\cdot d\mathbf{s}_0$$

$$-(\mathbf{G}+\mathbf{u}_0\times\nabla\mu)\cdot d\mathbf{s}_0\}(\Omega_i\tau_A\delta)^{-1}, \tag{18d}$$

where the vector function $\mathbf{G}$ is defined in Eq. (B13). Additional terms on the RHS of Eqs. (18c) and (18d), arising from the so far undetermined possible nonzero integration functions $g(\Psi_0)$ and $h(\Psi_0)$ from Eqs. (B10) and (B14), are, for simplicity, not considered here.

After substituting from Eqs. (18a)–(18c) into Eq. (17), and performing the time average over the Alfvén time (τ_A), we arrive at

$$\left\langle \frac{\partial K^\psi}{\partial t}\bigg|_{\text{CMHD}} \right\rangle \tau_A=\left\langle \oint \left[\mathbf{A}_0\times\left(\frac{\partial \mathbf{A}_0}{\partial t_2}+\frac{\partial \mathbf{A}_1}{\partial t_1}+\frac{\partial \mathbf{A}_2}{\partial t_0}\right)\right.\right.$$

$$\left.-3(\Omega_i\tau_A\delta)^{-1}(\mathbf{A}_0\cdot\mathbf{B}_0)(\mathbf{u}_1-\mu\mathbf{B}_1)\right]$$

$$\cdot d\mathbf{s}_0-3\oint (\mathbf{A}_0\cdot\mathbf{B}_0)[(\mathbf{j}_1-\lambda\mathbf{B}_1)\cdot d\mathbf{s}_0]$$

$$\left.\times(\Omega_i\tau_A\delta)^{-1}\cdot d\mathbf{s}_0\right\rangle_{\tau_A}\neq 0. \tag{19}$$

For the alternative case (18d) one has to replace in the last integral on the RHS of Eq. (19) the expression $(\mathbf{j}_1-\lambda\mathbf{B}_1)$ by the one $(\mathbf{G}+\mathbf{u}_0\times\nabla\mu)$, which is related to the compressibility of the plasma (cf. Appendix B).

Thus, with Eq. (19) we have arrived at the conclusion that the local K^ψ is on the CMHD time scale no longer an invariant of motion, where the violation is essentially due to the first-order electric field, the radial first-order fluxes of both the magnetic field and plasma particles and, moreover, due to the compressibility of the plasma. Since the zeroth-order quantities are not assumed to be on the IMHD time scale in equilibrium, it follows that the RHS of Eq. (19) also contains turbulent contributions.

Now, we investigate the time evolution of the global magnetic helicity integral K. The next order in our expansion scheme (8) yields, for the time derivative on the collision-MHD time scale (CMHD), the expression

$$\frac{\partial K}{\partial t}\bigg|_{\text{CMHD}}=\int \left[\mathbf{A}_0\cdot\left(\frac{\partial \mathbf{B}_0}{\partial t_2}+\frac{\partial \mathbf{B}_1}{\partial t_1}+\frac{\partial \mathbf{B}_2}{\partial t_0}\right)+\mathbf{B}_0\cdot\left(\frac{\partial \mathbf{A}_0}{\partial t_2}\right.\right.$$

$$\left.\left.+\frac{\partial \mathbf{A}_1}{\partial t_1}+\frac{\partial \mathbf{A}_2}{\partial t_0}\right)\right]d\tau+\int \left[\mathbf{A}_1\cdot\left(\frac{\partial \mathbf{B}_0}{\partial t_1}+\frac{\partial \mathbf{B}_1}{\partial t_0}\right)\right.$$

$$\left.+\mathbf{B}_1\cdot\left(\frac{\partial \mathbf{A}_0}{\partial t_1}+\frac{\partial \mathbf{A}_1}{\partial t_0}\right)\right]d\tau. \tag{20}$$

With the help of the dimensionless Maxwell's equations, Eq. (14) of Part I, Eq. (20) can be brought into the form

$$\frac{\partial K}{\partial t}\bigg|_{\text{CMHD}}=(\Omega_i\tau_A\delta)^{-1}\left(\oint \mathbf{A}_0\times\mathbf{E}_1\cdot d\mathbf{S}\right.$$

$$\left.-2\int \mathbf{B}_0\cdot\mathbf{E}_1\,d\tau\right)+\int \mathbf{B}_0\cdot\nabla X_1\,d\tau$$

$$+(\Omega_i\tau_A\delta)^{-1}\left(\oint \mathbf{A}_1\times\mathbf{E}_0\cdot d\mathbf{S}\right.$$

$$\left.-2\int \mathbf{B}_1\cdot\mathbf{E}_0\,d\tau\right)+\int \mathbf{B}_1\cdot\nabla X_0\,d\tau. \tag{21}$$

Based on the above-mentioned assumptions that the plasma is in its zeroth-order equilibrium, together with the boundary conditions, $\mathbf{B}_n\cdot d\mathbf{S}=\mathbf{j}_n\cdot d\mathbf{S}=\mathbf{u}_n\cdot d\mathbf{S}=0$ at the perfectly conducting wall, we arrive at

$$\frac{\partial K}{\partial t}\bigg|_{\text{CMHD}}=(\Omega_i\tau_A\delta)^{-1}\left(\oint \mathbf{A}_0\times\mathbf{E}_1\cdot d\mathbf{S}-2\int \mathbf{B}_0\cdot\mathbf{E}_1\,d\tau\right)$$

$$+\int \mathbf{B}_0\cdot\nabla X_1\,d\tau=\int \mathbf{B}_0\cdot\nabla X_1\,d\tau. \tag{22}$$

If we take for the multivalued gauge potential X_1, the expressions shown in (D4), then we obtain

$$\frac{\partial K}{\partial t}\bigg|_{\text{CMHD}}=\int \mathbf{B}_0\cdot\nabla\kappa\,\frac{\partial\beta}{\partial t_1}\,d\tau+\int \mathbf{B}_0\cdot\nabla\Phi\,d\tau\oint\left(\frac{\partial \mathbf{A}_0}{\partial t_2}\right.$$

$$\left.+\frac{\partial \mathbf{A}_1}{\partial t_1}+\frac{\partial \mathbf{A}_2}{\partial t_0}\right)\cdot\mathbf{e}_\phi\,d\Phi'+\int \mathbf{B}_0\cdot\nabla\theta\,d\tau$$

$$\times\oint_{\phi=0}\left(\frac{\partial \mathbf{A}_0}{\partial t_2}+\frac{\partial \mathbf{A}_1}{\partial t_1}+\frac{\partial \mathbf{A}_2}{\partial t_0}\right)\cdot\mathbf{e}_\theta\,d\theta'. \tag{23}$$

Since $\kappa(\mathbf{x})$ is a single-valued function, it follows that the first term on the RHS vanishes. Under the assumption that the first-order quantities have a harmonic time dependence on t_1, the performance of the time average over the Alfvén time scale τ_A will lead to

$$\left\langle \frac{\partial K}{\partial t}\bigg|_{\mathrm{CMHD}} \right\rangle_{\tau_A} = \left\langle \int \mathbf{B}_0 \cdot \nabla \Phi \, d\tau \oint_{\theta=0} \frac{\partial \mathbf{A}_0}{\partial t_2} \cdot \mathbf{e}_\phi \, d\Phi' + \int \mathbf{B}_0 \cdot \nabla \theta \, d\tau \oint_{\phi=0} \frac{\partial \mathbf{A}_0}{\partial t_2} \cdot \mathbf{e}_\theta \, d\theta' \right\rangle_{\tau_A}$$

$$= \left\langle \Psi_t \oint_{\theta=0} \frac{\partial \mathbf{A}_0}{\partial t_2} \cdot \mathbf{e}_\phi \, d\Phi' + \Psi_p \oint_{\phi=0} \frac{\partial \mathbf{A}_0}{\partial t_2} \cdot \mathbf{e}_0 \, d\theta' \right\rangle_{\tau_A}. \quad (24)$$

On account of the assumed perfectly conducting wall, there is a constant toroidal magnetic flux, so that Eq. (24) takes on the form

$$\left\langle \frac{\partial K}{\partial t}\bigg|_{\mathrm{CMHD}} \right\rangle_{\tau_A} = \left\langle \oint_{\phi=0} \mathbf{A}_0 \cdot \mathbf{e}_\theta \, d\theta' \oint_{\theta=0} \frac{\partial \mathbf{A}_0}{\partial t_2} \cdot \mathbf{e}_\phi \, d\Phi' \right\rangle_{\tau_A}. \quad (25)$$

The RHS of Eq. (25) arises from the multiple valuedness of the gauge potential X_1. These difficulties may be overcome by replacing K by the following modified form K_m:[3,4]

$$K_m = \int \mathbf{A} \cdot \mathbf{B} \, d\tau - \oint_{\phi=0} \mathbf{A} \cdot \mathbf{e}_\theta \, d\theta \oint_{\theta=0} \mathbf{A} \cdot \mathbf{e}_\phi \, d\Phi. \quad (26)$$

This modified global magnetic helicity integral K_m is then a gauge-independent invariant of motion on the MHD-collision time scale. This arises from the fact that, if the surrounding wall is a perfect conductor, then, in general, the second term in Eq. (26) can be assumed constant.[4]

V. THE TIME EVOLUTION OF THE HELICITY ON THE RDMHD TIME SCALE

Due to the resistivity on this time scale the magnetic field lines no longer preserve their topological properties, so it is meaningless to investigate K for each single flux tube. In this section we therefore investigate the time evolution of the global K only, taken over the whole plasma volume. The third order of our expansion scheme (8) (i.e., for $n=3$), yields for the time derivative on the resistive diffusion time scale the following expression:

$$\frac{\partial K}{\partial t}\bigg|_{\mathrm{RDMHD}} = \int \mathbf{A}_2 \cdot \left(\frac{\partial \mathbf{B}_0}{\partial t_1} + \frac{\partial \mathbf{B}_1}{\partial t_0} \right) d\tau + \int \mathbf{B}_2 \cdot \left(\frac{\partial \mathbf{A}_0}{\partial t_1} + \frac{\partial \mathbf{A}_1}{\partial t_0} \right) d\tau + \int \mathbf{A}_1 \cdot \left(\frac{\partial \mathbf{B}_0}{\partial t_2} + \frac{\partial \mathbf{B}_1}{\partial t_1} + \frac{\partial \mathbf{B}_2}{\partial t_0} \right) d\tau$$
$$+ \int \mathbf{B}_1 \cdot \left(\frac{\partial \mathbf{A}_0}{\partial t_2} + \frac{\partial \mathbf{A}_1}{\partial t_1} + \frac{\partial \mathbf{A}_2}{\partial t_0} \right) d\tau + \int \mathbf{A}_0 \cdot \left(\frac{\partial \mathbf{B}_0}{\partial t_3} + \frac{\partial \mathbf{B}_1}{\partial t_2} + \frac{\partial \mathbf{B}_2}{\partial t_1} + \frac{\partial \mathbf{B}_3}{\partial t_0} \right) d\tau$$
$$+ \int \mathbf{B}_0 \cdot \left(\frac{\partial \mathbf{A}_0}{\partial t_3} + \frac{\partial \mathbf{A}_1}{\partial t_2} + \frac{\partial \mathbf{A}_2}{\partial t_1} + \frac{\partial \mathbf{A}_3}{\partial t_0} \right) d\tau. \quad (27)$$

With the help of the dimensionless Maxwell's equations, Eq. (14) of Part I, we herefrom obtain

$$\frac{\partial K}{\partial t}\bigg|_{\mathrm{RDMHD}} = (\Omega_i \tau_A \delta)^{-1} \left(\oint \mathbf{A}_2 \times \mathbf{E}_0 \cdot d\mathbf{S} - 2 \int \mathbf{B}_2 \cdot \mathbf{E}_0 \, d\tau \right) + \int \mathbf{B}_2 \cdot \nabla X_0 \, d\tau + (\Omega_i \tau_A \delta)^{-1} \left(\oint \mathbf{A}_1 \times \mathbf{E}_1 \cdot d\mathbf{S} \right.$$
$$\left. - 2 \int \mathbf{B}_1 \cdot \mathbf{E}_1 \, d\tau \right) + \int \mathbf{B}_1 \cdot \nabla X_1 \, d\tau + (\Omega_i \tau_A \delta)^{-1} \left(\oint \mathbf{A}_0 \times \mathbf{E}_2 \cdot d\mathbf{S} - 2 \int \mathbf{B}_0 \cdot \mathbf{E}_2 \, d\tau \right) + \int \mathbf{B}_0 \cdot \nabla X_2 \, d\tau. \quad (28)$$

It may be shown that besides X_1, also the gauge potential X_2 has to be multivalued. The difficulties related to this multiple valuedness may again be removed by considering instead of the usual K also on the RDMHD time scale the modified helicity K_m, defined in Eq. (26). If we assume for simplicity a perfectly conducting wall, then those terms in Eq. (28) that contain the gradient of a gauge potential, will not contribute to K_m.[4] The application of the dimensionless second-order Ohm's law, Eq. (48) of Part I, to the remaining terms in Eq. (28), together with the boundary conditions $\mathbf{B}_n \cdot d\mathbf{S} = \mathbf{j}_n \cdot d\mathbf{S} = \mathbf{u}_n \cdot d\mathbf{S} = 0$, and the assumption that the plasma is during the evolution on the RDMHD time scale in direct contact with a perfectly conducting wall, leads to

$$\oint \mathbf{A}_1 \times \mathbf{E}_1 \cdot d\mathbf{S} = -\frac{1}{a_e} \left(\frac{\delta_e}{\delta_i} \right)^{1/2} \oint \mathbf{A}_1 \times \frac{\nabla p_{e0}}{n_{e0}} \cdot d\mathbf{S}$$
$$= 0 \quad \text{(polytropic plasma)}, \quad (29a)$$

Since the Jacobians J_0, J_1, and J_2 are supposed to be of order unity, $d\tau^{\psi}$ can be written in the envisaged form,

$$d\tau^{\psi} = d\tau_0 + \delta d\tau_1 + \delta^2 d\tau_2, \tag{A3}$$

where the volume elements $d\tau_0$, $d\tau_0 + \delta d\tau_1$, and $d\tau_0 + \delta d\tau_1 + \delta^2 d\tau_2$ are the volume elements enclosed between two corresponding neighboring magnetic surfaces. Similarly, also the infinitesimal surface elements $d\mathbf{s}^{\psi}$ can be written in the form

$$\begin{aligned} d\mathbf{s}^{\psi} &= [\nabla(\Psi_0 + \delta\Psi_1 + \delta^2\Psi_2)\cdot\nabla\vartheta\times\nabla\Phi]^{-1} \\ &\quad \times \nabla(\Psi_0 + \delta\Psi_1 + \delta^2\Psi_2)d\theta\, d\Phi \\ &= J_0\{\nabla\Psi_0 + \delta(\nabla\Psi_1 - J_1^{-1}\,\nabla\Psi_0) + \delta^2[\nabla\Psi_2 \\ &\quad - J_1^{-1}\,\nabla\Psi_1 + (J_0 J_1^{-2} - J_2^{-1})\nabla\Psi_0]\}d\theta\, d\Phi. \end{aligned} \tag{A4}$$

Thus, also the infinitesimal surface element $d\mathbf{s}^{\psi}$ can be written in this envisaged expansion form as

$$d\mathbf{s}^{\psi} = d\mathbf{s}_0 + \delta\, d\mathbf{s}_1 + \delta^2\, d\mathbf{s}_2. \tag{A5}$$

APPENDIX B: THE TIME EVOLUTION OF THE MAGNETIC SURFACES

The defining equation for the magnetic flux surface, with surface label $\Psi = \Psi_0 + \delta\Psi_1 + \delta^2\Psi_2 = \text{const}$, is given by

$$\mathbf{B}\cdot\nabla\Psi = (\mathbf{B}_0 + \delta\mathbf{B}_1 + \delta^2\mathbf{B}_2)\cdot\nabla(\Psi_0 + \delta\Psi_1 + \delta^2\Psi_2) = 0. \tag{B1}$$

It simply means that the magnetic field $\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}_1 + \delta^2\mathbf{B}_2$ lies in the flux surface $\Psi = \Psi_0 + \delta\Psi_1 + \delta^2\Psi_2 = \text{const}$ everywhere. Equation (B1) determines the flux function $\Psi(\mathbf{x})$ at a given instant. In an ideal MHD plasma, the magnetic field lines are frozen to the plasma; it means that the topology of the magnetic field cannot change relative to the plasma, so that we can ignore the time evolution of the flux surfaces. In a nonideal plasma, the situation is somewhat different. The finite resistivity allows magnetic-field diffusion to take place, while it also gives rise to transport processes, such as classical and neoclassical particle diffusion fluxes across the flux surfaces. This leads to a self-consistent evolution of the magnetic structure of the plasma.[7] Thus, the purpose of this appendix is to investigate the dynamical behavior of the magnetic flux surfaces over the different time scales.

Under the assumption $\Psi = \Psi(\mathbf{x}, t_0, t_1, t_2, t_3)$, the application of the multiple time scale derivative expansion scheme Eq. (13) of Part I, to Eq. (B1) leads to

$$\frac{\partial}{\partial t}(\mathbf{B}\cdot\nabla\Psi) = \sum_{n=0} \delta^n \sum_{m=0}^{n} \frac{\partial}{\partial t_m}\left(\sum_{s=0}^{n-m} \mathbf{B}_s\cdot\nabla\Psi_{n-s}\right). \tag{B2}$$

Thus, for each order of δ we have a separate equation for the time evolution of the magnetic surfaces.

δ^0: *Zeroth-order time evolution equation.* The zeroth-order equation describes the time evolution of the magnetic surfaces on the ion-gyration period time scale. From Eq. (B2) it follows that

$$\frac{\partial}{\partial t_0}(\mathbf{B}_0\cdot\nabla\Psi_0) = 0, \quad \text{leading to} \quad \frac{\partial\Psi_0}{\partial t_0} = 0. \tag{B3}$$

δ^1: *First-order time evolution equation (on the* IMHD *time scale).* For $n=1$, the expansion scheme (B2) yields

$$\frac{\partial}{\partial t_1}(\mathbf{B}_0\cdot\nabla\Psi_0) + \frac{\partial}{\partial t_0}(\mathbf{B}_0\cdot\nabla\Psi_1 + \mathbf{B}_1\cdot\nabla\Psi_0) = 0. \tag{B4a}$$

With the help of Maxwell's equations, Eq. (14) of Part I, and the zeroth-order Ohm's law, Eq. 19 of Part I, we herefrom obtain

$$\mathbf{B}_0\cdot\nabla\left(\frac{\partial\Psi_0}{\partial t_1} + \frac{\partial\Psi_1}{\partial t_0}\right) - (\Omega_i\tau_A\delta)^{-1}\nabla\cdot[\nabla\Psi_0\times(\mathbf{u}_0\times\mathbf{B}_0)] = 0,$$

and, furthermore, with the application of Eq. (B1),

$$\mathbf{B}_0\cdot\nabla\left[\left(\frac{\partial\Psi_0}{\partial t_1} + \frac{\partial\Psi_1}{\partial t_0}\right) + (\Omega_i\tau_A\delta)^{-1}(\mathbf{u}_0\cdot\nabla\Psi_0)\right] = 0, \tag{B4b}$$

leading to

$$\frac{\partial\Psi_0}{\partial t_1} + \frac{\partial\Psi_1}{\partial t_0} = -(\Omega_i\tau_A\delta)^{-1}(\mathbf{u}_0\cdot\nabla\Psi_0) + f(\Psi_0), \tag{B4c}$$

where the integration flux function $f(\Psi_0)$ has to be put to zero, as shown in Ref. 7, p. 192. Since the zeroth-order quantities do not depend on t_0, it follows that also Ψ_1 must be independent of t_0 in order to avoid secular terms.

Hence, we obtain from Eq. (B4c), the differential equation for the motion of the magnetic surfaces on the IMHD time scale in the following dimensional form:

$$\left.\frac{\partial\Psi}{\partial t}\right|_{\text{IMHD}} = -\mathbf{u}_0\cdot\nabla\Psi_0, \quad \text{i.e.,} \quad \frac{d\Psi}{dt} = 0,$$
$$\text{with} \quad \left(\frac{d}{dt} := \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right). \tag{B5}$$

Equation (B5) implies that during the evolution of the plasma over the IMHD time scale, the plasma and the magnetic flux surfaces move together, in other words, that the field lines are frozen to the plasma.

δ^2: *Second-order time evolution equation (on the CMHD time scale).* For $n=2$, the expansion scheme (B2) yields

$$\begin{aligned} &\frac{\partial}{\partial t_2}(\mathbf{B}_0\cdot\nabla\Psi_0) + \frac{\partial}{\partial t_1}(\mathbf{B}_0\cdot\nabla\Psi_1 + \mathbf{B}_1\cdot\nabla\Psi_0) + \frac{\partial}{\partial t_0}(\mathbf{B}_0\cdot\nabla\Psi_2 \\ &\quad + \mathbf{B}_1\cdot\nabla\Psi_1 + \mathbf{B}_2\cdot\nabla\Psi_0) = 0. \end{aligned} \tag{B6a}$$

With the help of Eq. (B4c) and Maxwell's equations Eq. (14) of Part I, we herefrom obtain

$$\begin{aligned} &\mathbf{B}_0\cdot\nabla\left(\frac{\partial\Psi_0}{\partial t_2} + \frac{\partial\Psi_1}{\partial t_1} + \frac{\partial\Psi_2}{\partial t_0}\right) - [\mathbf{B}_1\cdot\nabla(\mathbf{u}_0\cdot\nabla\Psi_0) \\ &\quad + \nabla\Psi_0\cdot\nabla\times\mathbf{E}_1 + \nabla\Psi_1\cdot\nabla\times\mathbf{E}_0](\Omega_i\tau_A\delta)^{-1} = 0, \end{aligned} \tag{B6b}$$

and after expressing $\mathbf{E}_0$ and $\mathbf{E}_1$ by the zeroth- and first-order Ohm's law, Eqs. (14) and (24) of Part I, we arrive at

$$
(\Omega_i\tau_A\delta)\mathbf{B}_0\cdot\nabla\left(\frac{\partial\Psi_0}{\partial t_2}+\frac{\partial\Psi_1}{\partial t_1}+\frac{\partial\Psi_2}{\partial t_0}\right)-[\mathbf{B}_1\cdot\nabla(\mathbf{u}_0\cdot\nabla\Psi_0)]
$$

$$
=-\left\{\nabla\Psi_0\cdot\nabla\times\left[\mathbf{u}_1\times\mathbf{B}_0+\mathbf{u}_0\times\mathbf{B}_1+(\omega_e\tau_A)\left(\frac{\delta_e}{\delta_i}\right)^{1/2}\right.\right.
$$

$$
\left.\times\frac{\nabla p_{e0}}{n_{e0}}-(\Omega_i\tau_A n_0)^{-1}(\mathbf{j}_0\times\mathbf{B}_1+\mathbf{j}_1\times\mathbf{B}_0)\right]
$$

$$
\left.+\nabla\Psi_1\cdot\nabla\times(\mathbf{u}_0\times\mathbf{B}_0)\right\}. \tag{B6c}
$$

Equation (B6c) may be further simplified by taking into account the following two assumptions: (1) The plasma behaves polytropically along the field lines, i.e., $(\nabla p_{e0}/n_{e0})\simeq\nabla f(n_{e0})$. (2) The plasma has relaxed to its zeroth-order equilibrium state, which is defined by

$$
\mathbf{j}_0=\lambda(\mathbf{x};t_1,t_2,t_3)\mathbf{B}_0 \quad \text{and} \quad \mathbf{u}_0=\mu(\mathbf{x};t_1,t_2,t_3)\mathbf{B}_0. \tag{B7}
$$

Since the term $\nabla p_{e0}/n_{e0}$ is due to the first-order Ohm's law valid for the IMHD time scale, where the adiabatic law applies, assumption (1) means no restriction. Furthermore, according to our multiple time scale procedure, one has for the preceding time scales either to take the asymptotic limit $t\rightarrow\infty$ or to perform the time averages. Thus, the first of the equations (B7) also means no restriction. The only limitation is the one appearing in the second equation (B7), stating that the perpendicular rotation is of order δ^1, an assumption which is true for present-day large fusion devices. With the help of these assumptions Eq. (B6c) can be rewritten in the form

$$
\mathbf{B}_0\cdot\nabla\left(\frac{\partial\Psi_0}{\partial t_2}+\frac{\partial\Psi_1}{\partial t_1}+\frac{\partial\Psi_2}{\partial t_0}\right)
$$

$$
=-(\mathbf{B}_0\cdot\nabla(\mathbf{u}_1\cdot\nabla\Psi_0-\mu\mathbf{B}_1\cdot\nabla\Psi_0)+\{\nabla\Psi_0\cdot\nabla
$$

$$
\times[(\Omega_i\tau_A n_0)^{-1}\times(\mathbf{j}_0\times\mathbf{B}_1+\mathbf{j}_1\times\mathbf{B}_0)]\})(\Omega_i\tau_A\delta)^{-1}, \tag{B6d}
$$

leading to

$$
\frac{\partial\Psi_0}{\partial t_2}+\frac{\partial\Psi_1}{\partial t_1}+\frac{\partial\Psi_2}{\partial t_0}=-\nabla\Psi_0\cdot[(\mathbf{u}_1-\mu\mathbf{B}_1)
$$

$$
+(\Omega_i\tau_A n_0)^{-1}(\mathbf{j}_1-\lambda\mathbf{B}_1)]
$$

$$
\times(\Omega_i\tau_A\delta)^{-1}+g(\Psi_0), \tag{B8}
$$

where $g(\Psi_0)$ is an arbitrary flux function. Contrary to the IMHD case of Eq. (B4c)—this integration function on the CMHD time scale needs not necessarily to be zero. Under the assumption that the first-order quantities have a harmonic t_0 and t_1 dependence, we obtain, after performing the time averages over the time scales τ_0 and τ_A,

$$
\left\langle\frac{\partial\Psi_0}{\partial t_2}\right\rangle_{\tau_0,\tau_A}=-\langle[(\mathbf{u}_1-\mu\mathbf{B}_1)\cdot\nabla\Psi_0+(\Omega_i\tau_A n_0)^{-1}(\mathbf{j}_1
$$

$$
-\lambda\mathbf{B}_1)\cdot\nabla\Psi_0](\Omega_i\tau_A\delta)^{-1}+g(\Psi_0)\rangle_{\tau_0,\tau_A}. \tag{B9}
$$

Finally Eq. (B9) can be brought into the following dimensional form:

$$
\left.\frac{\partial\Psi}{\partial t}\right|_{\text{CMHD}}
$$

$$
=-\delta[(\mathbf{u}_1-\mu\mathbf{B}_1)\cdot\nabla\Psi_0+n_0^{-1}(\mathbf{j}_1-\lambda\mathbf{B}_1)\cdot\nabla\Psi_0)]
$$

$$
=-[\mathbf{u}\cdot\nabla\Psi+\delta n_0^{-1}(\mathbf{j}_1-\lambda\mathbf{B}_1)\cdot\nabla\Psi_0]
$$

$$
+g(\Psi_0)+O(\delta^2), \tag{B10a}
$$

or, equivalently,

$$
\frac{d\Psi}{dt}=-\delta n_0^{-1}(\mathbf{j}_1-\lambda\mathbf{B}_1)\cdot\nabla\Psi_0+g(\Psi_0)+O(\delta^2), \tag{B10b}
$$

where $\mathbf{u}=\mathbf{u}_0+\delta\mathbf{u}_1$ and $\Psi=\Psi_0+\delta\Psi_1$.

Equations (B10), which describe the time evolution of the magnetic surfaces on the CMHD time scale, can be brought into another equivalent form in the following sense. By inserting from the dimensionless one fluid momentum equation from the IMHD time scale (Part I), Eq. (B6d) can be written in the following form:

$$
\mathbf{B}_0\cdot\nabla\left(\frac{\partial\Psi_0}{\partial t_2}+\frac{\partial\Psi_1}{\partial t_1}+\frac{\partial\Psi_2}{\partial t_0}\right)
$$

$$
=-\left\{\mathbf{B}_0\cdot\nabla(\mathbf{u}_1\cdot\nabla\Psi_0-\mu\mathbf{B}_1\cdot\nabla\Psi_0)\right.
$$

$$
\left.-\left[\nabla\Psi_0\cdot\nabla\times\left(\frac{\partial\mathbf{u}_0}{\partial t_1}+(\mathbf{u}_0\cdot\nabla)\mathbf{u}_0\right)\right]\right\}(\Omega_i\tau_A\delta)^{-1}. \tag{B11}
$$

Since $\mathbf{u}_0=\mu(\mathbf{x};t_1,t_2,t_3)\mathbf{B}_0$, Eq. (B11) can also be rewritten in the following way:

$$
\mathbf{B}_0\cdot\nabla\left(\frac{\partial\Psi_0}{\partial t_2}+\frac{\partial\Psi_1}{\partial t_1}+\frac{\partial\Psi_2}{\partial t_0}\right)
$$

$$
=-\{\mathbf{B}_0\cdot\nabla[\mathbf{u}_1\cdot\nabla\Psi_0-\mu\mathbf{B}_1\cdot\nabla\Psi_0-\mathbf{G}\cdot\nabla\Psi_0
$$

$$
-(\mathbf{u}_0\times\nabla\mu\cdot\nabla\Psi_0)]\}(\Omega_i\tau_A\delta)^{-1}, \tag{B12}
$$

where the vector function $\mathbf{G}(\mathbf{x};t_1,t_2,t_3)$ is defined by

$$
\nabla(\mathbf{G}\cdot\nabla\Psi_0)=\nabla\Psi_0\times\nabla\left(\frac{\partial\mu}{\partial t_1}\right). \tag{B13}
$$

Under the assumption that the first- and second-order quantities show a harmonic dependence on t_0 and t_1, we obtain, after performing the time averages over the second-order time evolution equation from Eq. (B12):

$$
\left\langle\frac{\partial\Psi_0}{\partial t_2}\right\rangle_{\tau_0,\tau_A}=-\langle[(\mathbf{u}_1-\mu\mathbf{B}_1)\cdot\nabla\Psi_0-\mathbf{G}\cdot\nabla\Psi_0
$$

$$
-(\mathbf{u}_0\times\nabla\mu\cdot\nabla\Psi_0)](\Omega_i\tau_A\delta)^{-1}
$$

$$
+h(\Psi_0)\rangle_{\tau_0,\tau_A}, \tag{B14}
$$

where $h(\Psi_0)$ is again an arbitrary, not necessarily nonzero, flux function. Furthermore, Eq. (B14) can be written in the following dimensional form:

$$\left.\frac{\partial\Psi}{\partial t}\right|_{\text{CMHD}} = -[\delta(\mathbf{u}_1-\mu\mathbf{B}_1)\cdot\nabla\Psi_0-\mathbf{G}\cdot\nabla\Psi_0 - (\mathbf{u}_0\times\nabla\mu\cdot\nabla\Psi_0)]+h(\Psi_0), \tag{B15a}$$

or, alternatively,

$$\frac{d\Psi}{dt}=[\mathbf{G}\cdot\nabla\Psi_0+(\mathbf{u}_0\times\nabla\mu\cdot\nabla\Psi_0)]+h(\Psi_0)+O(\delta^2). \tag{B15b}$$

Equation (B15a) shows that there are two different cases for the time evolution of the magnetic surfaces on the CMHD time scale.

Case 1: If the plasma is assumed to be behave like an *incompressible* fluid,

$$\nabla\cdot\mathbf{u}_0=\nabla\cdot[\mu(\mathbf{x};t)\mathbf{B}_0]=\mathbf{B}_0\cdot\nabla\mu=0,$$

$$\text{i.e.,}\quad \nabla\mu\uparrow\uparrow\nabla\Psi_0 \quad\text{and}\quad \left\langle\nabla\left(\frac{\partial\mu}{\partial t_1}\right)\right\rangle_{\tau_0,\tau_1}=0,$$

$$\left.\frac{\partial\Psi}{\partial t}\right|_{\text{CMHD}}+\delta(\mathbf{u}_1-\mu\mathbf{B}_1)\cdot\nabla\Psi_0=\left.\frac{d\Psi}{dt}\right|_{\text{CMHD}}=h(\Psi_0). \tag{B16a}$$

Case 2: If the plasma is assumed as a *compressible* fluid, then we obtain

$$\left.\frac{\partial\Psi}{\partial t}\right|_{\text{CMHD}}+\delta(\mathbf{u}_1\cdot\nabla\Psi_0-\mu\mathbf{B}_1\cdot\nabla\Psi_0) =\left.\frac{d\Psi}{dt}\right|_{\text{CMHD}}=h(\Psi_0)+\delta(\mathbf{G}+\mathbf{u}_0\times\nabla\mu)\cdot\nabla\Psi_0. \tag{B16b}$$

From Eqs. (B10b), (B16a), and (B16b), we infer that on the MHD-collision time scale, the magnetic field lines are no longer frozen to the plasma, essentially due to the radial components of the first-order magnetic field and the first-order current or equivalently due to the compressibility. Thus, it is obvious that the compressibility not only affects the particle density, but also the magnetic field configuration.

The important key question to be answered remains, however, which processes may lead to such a decoupling between the plasma and the field lines. To answer this question, one has to keep in mind that these processes must fulfill the following requirements. First, they have to evolve on a time scale, which is of the order of the CMHD time scale. Second, these processes need the compressibility for their development, and do not depend on the resistivity. In this sense, it was shown that the driven magnetic reconnection process is the best candidate on the CMHD time scale.[8,9]

APPENDIX C: THE TIME EVOLUTION OF THE VOLUME ELEMENT $d\tau^{\psi}$

In this appendix the time evolution of the volume element $d\tau^{\psi}$ will be calculated in the following sense. The notation $\partial d\tau/\partial t=\partial d\tau/\partial\Psi\cdot\partial\Psi/\partial t$, with $\partial d\tau/\partial\Psi=J\,d\vartheta\,d\Phi$ stands as a shorthand writing symbolically, therefore that instead of the volume integral one has only to perform the integration over the angular variables ϑ and Φ at the surface Ψ=const. We then formally have to deal with

$$\frac{\partial d\tau}{\partial t}=\frac{\partial d\tau}{\partial\Psi}\cdot\frac{\partial\Psi}{\partial t}=\frac{\partial d\tau_0}{\partial t_0}+\delta\left(\frac{\partial d\tau_0}{\partial t_1}+\frac{\partial d\tau_1}{\partial t_0}\right) +\delta^2\left(\frac{\partial d\tau_0}{\partial t_2}+\frac{\partial d\tau_1}{\partial t_1}+\frac{\partial d\tau_2}{\partial t_0}\right), \tag{C1}$$

with

$$\frac{\partial\Psi}{\partial t}=\frac{\partial\Psi_0}{\partial t_0}+\delta\left(\frac{\partial\Psi_0}{\partial t_1}+\frac{\partial\Psi_1}{\partial t_0}\right)+\delta^2\left(\frac{\partial\Psi_0}{\partial t_2}+\frac{\partial\Psi_1}{\partial t_1}+\frac{\partial\Psi_2}{\partial t_0}\right) \tag{C2}$$

and

$$\frac{\partial d\tau}{\partial\Psi}=J\,d\vartheta\,d\Phi=J_0\left[1-\delta\left(\frac{J_0}{J_1}\right)-\delta^2\left(\frac{J_0}{J_2}\right)\right]d\vartheta\,d\Phi. \tag{C3}$$

By employing Eqs. (B3), (B4c), (B10), (B15), and (B16), it can be shown that δ^0: *zeroth-order time evolution,*

$$\frac{\partial d\tau_0}{\partial t_0}=0; \tag{C4}$$

δ^1: *first-order time evolution (on the IMHD time scale),*

(Dimensionless form)

$$\frac{\partial d\tau_0}{\partial t_1}+\frac{\partial d\tau_1}{\partial t_0}=-(\Omega_i\tau_A\delta)^{-1}(\mathbf{u}_0\cdot d\mathbf{s}_0), \tag{C5a}$$

$$\text{(Dimensional form)}\left.\frac{\partial d\tau}{\partial t}\right|_{\text{IMHD}}=-\mathbf{u}_0\cdot d\mathbf{s}_0; \tag{C5b}$$

and δ^2: *second-order time evolution (on the CMHD time scale),*

(Dimensionless form)

$$\frac{\partial d\tau_0}{\partial t_2}+\frac{\partial d\tau_1}{\partial t_1}+\frac{\partial d\tau_2}{\partial t_0} =-[\mathbf{u}_1\cdot d\mathbf{s}_0-\mu\mathbf{B}_1\cdot d\mathbf{s}_0+(\Omega_i\tau_A n_0)^{-1} \times(\mathbf{j}_1-\lambda\mathbf{B}_1)\cdot d\mathbf{s}_0)](\Omega_i\tau_A\delta)^{-1}+J_0 g(\Psi_0)d\vartheta\,d\Phi, \tag{C6a}$$

or equivalently,

$$\frac{\partial d\tau_0}{\partial t_2}+\frac{\partial d\tau_1}{\partial t_1}+\frac{\partial d\tau_2}{\partial t_0}=-[\mathbf{u}_1\cdot d\mathbf{s}_0-\mu\mathbf{B}_1\cdot d\mathbf{s}_0-(\mathbf{G} +\mathbf{u}_0\times\nabla\mu)\cdot d\mathbf{s}_0](\Omega_i\tau_A\delta)^{-1} +J_0 h(\Psi_0)d\vartheta\,d\Phi; \tag{C6b}$$

and

(Dimensional form)

$$\left.\frac{\partial d\tau}{\partial t}\right|_{\text{CMHD}}=-\delta(\mathbf{u}_1\cdot d\mathbf{s}_0-\mu\mathbf{B}_1\cdot d\mathbf{s}_0)+\delta(\mathbf{j}_1-\lambda\mathbf{B}_1)\cdot d\mathbf{s}_0 +(\Omega_i\tau_A\delta)J_0 g(\Psi_0)d\vartheta\,d\Phi. \tag{C6c}$$

APPENDIX D: THE GAUGE POTENTIAL FUNCTION $X(\mathbf{x},t)$

To evaluate in our calculation the surface integral terms that contain the gauge potential function X, one has to be aware of the nature of X, since for a multiply connected region X needs not necessarily to be single valued. This appendix is devoted to the investigation of the nature of the gauge potential function X for different orderings. Let $\mathbf{w}(\mathbf{x};t)$ be any single-valued vector field, satisfying

$$\nabla\Psi\cdot(\nabla\times\mathbf{w})=0. \tag{D1}$$

Let $z(\Psi)$ be any particular point on each magnetic flux surface with label Ψ. For each point $\mathbf{x}$ in space, define

$$\nu(\mathbf{x},t)=\int_z^x d\mathbf{x}\cdot\mathbf{w}, \tag{D2}$$

where the path of integration lies on the magnetic flux surface Ψ=const. In view of (D1), it follows from Stoke's theorem that the value of $\nu(\mathbf{x},t)$ is independent of the path, joining $z(\Psi)$ to $\mathbf{x}$ for all paths continuously deformable into each other. However, not all paths are deformable into each other, so ν is a multivalued function. It can be written in the following general form:[10]

$$\nu(\mathbf{x};t)=\beta(t)\kappa(\mathbf{x})+\Phi\oint_{\substack{\theta=0\\ \psi=c}}\mathbf{w}\cdot\mathbf{e}_\phi\,d\Phi'+\theta\oint_{\substack{\phi=0\\ \psi=c}}\mathbf{w}\cdot\mathbf{e}_\theta\,d\theta', \tag{D3}$$

where $\kappa(\mathbf{x})$ is some single-valued function and the loop integrals are taken at the surfaces Ψ=const in the direction of increasing Φ and θ, respectively. One of the possible choices for $\mathbf{w}$ is $\mathbf{w}=\mathbf{A}$, where $\mathbf{A}$ is the magnetic vector potential. Accordingly, $\nu(\mathbf{x};t)$ can be written as follows:

$$\nu(\mathbf{x};t)=\beta(t)\kappa(\mathbf{x})+\Phi\Psi_p^h+\theta\Psi_t, \tag{D4a}$$

with

$$\Psi_p^h=\oint_{\substack{\theta=0\\ \psi=c}}\mathbf{A}\cdot\mathbf{e}_\phi\,d\Phi' \tag{D4b}$$

and

$$\Psi_t=\oint_{\substack{\phi=0\\ \psi=c}}\mathbf{A}\cdot\mathbf{e}_\theta\,d\theta'. \tag{D4c}$$

Here Ψ_p^h is the poloidal magnetic flux through the hole of the toroidal flux surface Ψ=const and Ψ_t is the corresponding toroidal flux within that surface. By considering at the moment arbitrary gauge $\nabla\nu$ for the magnetic vector potential $\mathbf{A}$, it follows that $\mathbf{A}$ can be replaced by $\mathbf{A}'=\mathbf{A}-\nabla\nu$, or equivalently $\partial\mathbf{A}/\partial t$ by $\partial\mathbf{A}'/\partial t=\partial\mathbf{A}/\partial t-\nabla X$, i.e., $X=\partial\nu/\partial t$. According to our multiple time scale scheme Eq. (13) of Part I, and the dimensionless Maxwell's equations, Eq. (14) of Part I, the dimensionless Faraday's law becomes

$$\sum_{n=0}\delta^n\mathbf{E}_n=-\sum_{n=0}\delta^n\left[(\Omega_i\tau_A\delta)\left(\sum_{m=0}^{n+1}\frac{\partial}{\partial t_m}(\mathbf{A})_{n-m+1}\right)-\nabla X_n\right], \tag{D5a}$$

with

$$X_n=\kappa(\mathbf{x})\left(\sum_{m=0}^{n+1}\frac{\partial}{\partial t_m}\beta_{n+1-m}\right)+\Phi\left(\sum_{m=0}^{n+1}\frac{\partial}{\partial t_m}(\Psi_p^h)_{n+1-m}\right)+\theta\left(\sum_{m=0}^{n+1}\frac{\partial}{\partial t_m}(\Psi_t)_{n+1-m}\right). \tag{D5b}$$

The poloidal magnetic flux Ψ_p^h through the torus hole is related to our poloidal flux Ψ, by $\Psi=\Psi_p^{\mathrm{tot}}-\Psi_p^h$, where Ψ_p^{tot} is the total poloidal flux through the torus hole within the magnetic axis. Here, we have to take into consideration the following relation:

$$\frac{\partial\Psi_p^{\mathrm{tot}}}{\partial t}=-(\Omega_i\tau_A\delta)^{-1}\oint_{\mathrm{axis}}\mathbf{E}\cdot d\mathbf{l}, \tag{D6}$$

where the path of integration lies on the magnetic axis defined by $\nabla\Psi=0$. After performing the above multiple time scale scheme, we obtain for each order of δ a separate equation for the corresponding gauge potential X_n.

δ^0: Zeroth-order gauge potential function X_0. Taking into account the fact that the time derivatives $\partial/\partial t_0$ vanish, the zeroth-order gauge potential X_0 is determined by

$$X_0=\kappa(\mathbf{x})\frac{\partial\beta_0}{\partial t_1}-\Phi(\Omega_i\tau_A\delta)^{-1}\oint_{\mathrm{axis}}\mathbf{E}_0\cdot d\mathbf{l}_0+\left(\theta\frac{d\Psi_t}{d\Psi}-\Phi\right)\frac{\partial\Psi_0}{\partial t_1}, \tag{D7}$$

where the path of integration in this case lies on the magnetic axis $\nabla\Psi_0=0$. From Eq. (B4c) and the fact that the plasma has relaxed to its zeroth-order equilibrium, (i.e., $\mathbf{u}_0=\mu\mathbf{B}_0$ and $\mathbf{j}_0=\lambda\mathbf{B}_0$), follows that X_0 is a single-valued function.

δ^1: First-order gauge potential function X_1. Along the same lines as for the zeroth-order case, the corresponding expression for the first-order gauge potential X_1 reads similarly,

$$X_1=\kappa(\mathbf{x})\frac{\partial\beta_0}{\partial t_1}-\Phi(\Omega_i\tau_A\delta)^{-1}\oint_{\mathrm{axis}}\mathbf{E}\cdot d\mathbf{l}+\left(\theta\frac{d\Psi_t}{d\Psi}-\Phi\right)\times\left(\frac{\partial\Psi_0}{\partial t_2}+\frac{\partial\Psi_0}{\partial t_2}\right), \tag{D8}$$

where, $\mathbf{E}=\mathbf{E}_0+\delta\mathbf{E}_1$, $d\mathbf{l}=d\mathbf{l}_0+\delta d\mathbf{l}_1$, and the path of integration lies on the magnetic axis defined by $\nabla\Psi=\nabla\Psi_0+\delta\nabla\Psi_1=0$. After performing the time average over the IMHD time scale τ_A, and using Eq. (B9) together with the first-order dimensionless Ohm's law, Eq. (24) of Part I, it can be shown that both the second and the third terms on the RHS of Eq. (D8) do not vanish. Thus, we conclude that the first-order gauge potential X_1 is a multiple-valued function.

Analogously, we may infer that also the second-order gauge potential X_2 is a multiple-valued function.

[1] J. B. Taylor, Phys. Rev. Lett. **33**, 1139 (1974).

[2] J. B. Taylor, *Proceedings of the 3rd Topical Conference on Pulsed High Beta Plasmas*, Culham Laboratory, United Kingdom, 1975, edited by D. E. Evans (Pergamon, Oxford, 1976), p. 59.

[3] M. K. Bevir, C. G. Gimblett, and G. Miller, Phys. Fluids **28**, 1826 (1985).

[4] J. B. Taylor, Rev. Mod. Phys. **58**, 741 (1986).

[5] P. G. M. Centen, Ph.D. thesis, University of Eindhoven, The Netherlands, 1984.

[6] L. Woltjer, Rev. Mod. Phys. **32**, 914 (1960).

[7] W. D. D'haeseler, W. N. Hitchon, J. D. Callen, and J. L. Shohet, *Flux Coordinates and Magnetic Field Structure* (Springer-Verlag, Berlin, 1991).

[8] T. Sato and K. Kusano, Phys. Rev. Lett. **54**, 808 (1985).

[9] T. Sato, R. Horiuchi, and K. Kusano, *Proceedings of Contributed Papers*, 1986, International Conference on Controlled Fusion and Plasma Physics, Kyoto, Japan 1986 (International Atomic Energy Agency, Vienna, 1986), IAEA Paper No. CN-47/D-V-11.

[10] M. Kruskal and R. M. Kulsrud, Phys. Fluids **1**, 265 (1958).