J. Plasma Physics (1996), vol. 00, part 0, pp. 000–000

Transport equations on different time scales for intermediately and strongly collisional regimes

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(Received 16 January 1995 and in revised form 5 October 1995)

The plasma transport equations for a weakly collisional plasma have previously been derived for four different time scales. This paper is devoted to the derivation of the plasma transport equations for the two other complementary regimes: the intermediately collisional regime (ICR) (i.e. for the case where the transit time ω_α^{-1} is of the same order as the collision time ν_α^{-1}, $\omega_\alpha^{-1} \approx \nu_\alpha^{-1}$), and the strongly collisional regime (SCR) (i.e. for the case of $\omega_\alpha^{-1} \gg \nu_\alpha^{-1}$) for different time scales. It is shown that the lowest-order gyromotion is unperturbed by collisions. On the Alfvén time scale, one merely obtains for both the intermediately collisional case and the strongly collisional case the single-fluid ideal MHD equations, if certain additional requirements are satisfied. On the MHD-collision time scale, one arrives at the full set of transport equations, where in both cases, contrary to the weakly collisional case, no turbulent terms are found. On the resistive diffusion timescale, one ends up with the known transport equations, with the addition of turbulent contributions.

1. Introduction

In previous work (Edenstrasser 1995), a multiple-time-scale derivative expansion scheme was applied to the dimensionless Fokker–Planck equation and Maxwell equations, with the plasma transport equations being derived for the case of a strongly magnetized, weakly collisional plasma (the WCR) (i.e. for the case where the transit time ω_α^{-1} is much shorter than the collision time ν_α^{-1}, $\omega_\alpha^{-1} \ll \nu_\alpha^{-1}$) for different MHD time scales. The time scales considered were the ion-gyration, Alfvén, MHD-collision and resistive diffusion. It was shown that the solution of the zeroth-order equations (i.e. on the ion-gyration time scale) leads to force-free equilibria and to the ideal MHD Ohm's law. Furthermore, the solution of the first-order equations (i.e. on the Alfvén time scale) leads to the ideal MHD equations. On the MHD-collision time scale, the MHD transport equations were obtained, with additional turbulent terms, where the related transport quantities are one order larger in the expansion parameter than those of the classical theory. Finally, on the resistive diffusion time scale, the known transport equations, again with additional turbulent terms, were obtained.

Since these equations are valid for a high-temperature, weakly collisional plasma, we are motivated to investigate on the basis of this multiple-time-scale approach also the transport equations for the two other limiting regimes, namely the intermediately collisional regime (ICR, i.e. $\omega_\alpha^{-1} \approx \nu_\alpha^{-1}$), and the

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strongly collisional regime (SCR, i.e. $\omega_\alpha^{-1} \gg \nu_\alpha^{-1}$) ones. These investigations are the subject of this paper. In this connection there also arises the question of the range of validity of the ideal MHD equations for the different regimes, a problem that will be discussed in a forthcoming publication.

The paper is organized as follows. In §2, the scheme of derivation is comprehensively reviewed. In §§3 and 4, the plasma transport equations for an intermediately collisional plasma and a strongly collisional one respectively are presented. In §5, a brief summary and the relevant conclusions are presented.

2. The scheme of derivation

Apart from some minor changes, the plasma transport equations are derived in §§3 and 4 in nearly the same way as in the case of a weakly collisional plasma (Edenstrasser 1995). In order to avoid too much repetition, in this section we explain step by step the lines along which the transport equations are derived.

First of all, one has to keep in mind that there are two types of multiple-time-scale expansions. The first is the species-dependent multiple-time-scale ordering in kinetic theory, and the second is the multiple-timescale ordering in magnetohydrodynamic (MHD) theory, including also the electromagnetic field quantities. However, these multiple-time-scale expansion schemes are not independent of each other, since on going from the kinetic to the fluid description, the expansion schemes from electrons and ions must merge together into the MHD expansion scheme. Therefore in the derivation of the transport equations they are treated together in the following sense.

(i) In kinetic theory, the following multiple-time-scale orderings are assumed to hold:

(*a*) for the ICR ($\nu_\alpha^{-1} \approx \omega_\alpha^{-1}$),

with
$$\left.\begin{aligned} &\tau_{\alpha 0} := \Omega_\alpha^{-1} \ll \tau_{\alpha 1} := \omega_\alpha^{-1} \approx \nu_\alpha^{-1} \ll \tau_{\alpha 2} := \tau_{\alpha cd}, \\ &\nu_\alpha/\Omega_\alpha \approx \omega_\alpha/\Omega_\alpha := \delta_\alpha; \end{aligned}\right\} \tag{1}$$

(*b*) for the SCR ($\nu_\alpha^{-1} \ll \omega_\alpha^{-1}$),

with
$$\left.\begin{aligned} &\tau_{\alpha 0} := \Omega_\alpha^{-1} \ll \tau_{\alpha 1} := \nu_\alpha^{-1} \ll \tau_{\alpha 2} := \omega_\alpha^{-1} \ll \tau_{\alpha 3} := \tau_{\alpha cd}, \\ &\omega_\alpha/\nu_\alpha \approx \nu_\alpha/\Omega_\alpha := \delta_\alpha. \end{aligned}\right\} \tag{2}$$

Here Ω_α is the Larmor frequency, δ_α is the ratio of the Larmor radius (mean free path) to the hydromagnetic length for the ICR (SCR), and $\tau_{\alpha cd}$ is the classical diffusion time, for which we may take, for example, the cross-field heat-conduction time (Jardin 1985). The hydromagnetic length L is defined as the length over which the MHD variables change considerably (e.g. in the case of a fusion device, the plasma radius). The physical conditions under which a plasma satisfies the above time scale ordering, particularly those of the last ordering concerning the classical diffusion time, remain to be analysed. Furthermore, the unique time variable t then has to be expanded into three (four) formally independent time variables $t_{\alpha n}$ for the ICR (SCR), so that we obtain, in a standardized form,

$$t = \sum_{n=0} \delta_\alpha^n t_{\alpha n}, \quad \frac{\partial}{\partial t} = \sum_{n=0} \delta_\alpha^n \frac{\partial}{\partial t_{\alpha n}}. \tag{3}$$

(ii) The Fokker–Planck equation must be brought into dimensionless form by normalizing all physical quantities $\tilde{Q}$ with respect to some characteristic values (Edenstrasser 1995):

$$\left.\begin{aligned} t &= \frac{\tilde{t}}{\tau_{\alpha 0}}, \quad f_\alpha = \frac{\tilde{f}_\alpha v_{\rm th}^{\alpha 3}}{\bar{n}_\alpha}, \quad \mathbf{B} = \frac{\tilde{\mathbf{B}}}{\bar{B}}, \quad \mathbf{E} = \frac{\tilde{\mathbf{E}} c}{v_A \bar{B}}, \\ v &= \frac{\tilde{v}}{v_{\rm th}^{\alpha}}, \quad \nabla = L\tilde{\nabla}, \quad C = \frac{\tilde{C}_\alpha}{\bar{C}_\alpha}, \end{aligned}\right\} \tag{4a}$$

where α refers to the particle species. Here v_A is the Alfvén velocity, $\bar{n}_\alpha$ is the maximum particle density, $\bar{B}$ is the maximum magnetic field, and $v_{\rm th}^{\alpha}$ is the maximum thermal velocity, all values taken at a characteristic instant. Thus the dimensionless Fokker–Planck equation reads

$$\frac{\partial f_\alpha}{\partial t} + P(\delta_\alpha)\,\mathbf{v}\,.\,\nabla f_\alpha + \sigma(q_\alpha)\,(a_\alpha \mathbf{E} + \mathbf{v}\times\mathbf{B})\,.\,\frac{\partial f_\alpha}{\partial \mathbf{v}} = \Gamma_\alpha C_\alpha, \tag{4b}$$

where $P(\delta_\alpha)$ is defined as $P(\delta_\alpha) = \delta_\alpha$ for the ICR and $P(\delta_\alpha) = K\delta_\alpha^2$ for the SCR, with $K := \nu_\alpha/\delta_\alpha \Omega_\alpha = O(1)$. Furthermore, the following definitions are employed:

$$\sigma(q_\alpha) := \operatorname{sign}(q_\alpha), \qquad a_\alpha := \frac{v_A}{v_{\rm th}^{\alpha}} = (\omega_\alpha \tau_A)^{-1}. \tag{5a}$$

The dimensionless factors Γ_α in front of the collision operator, defined in Edenstrasser (1995), are given by

$$\Gamma_i := -\frac{3}{2^{\frac{3}{2}}}\frac{\nu_i}{\Omega_i}, \qquad \Gamma_e := -\frac{3}{4}\frac{\nu_e}{\Omega_e}, \tag{5b}$$

and in both cases are of order δ_α, i.e.

$$\Gamma_\alpha := \Lambda_\alpha \delta_\alpha = O(1)\,\delta_\alpha. \tag{5c}$$

(iii) In MHD theory, the following multiple-time-scale orderings are assumed to hold:

(*a*) for the ICR,

$$\tau_0 := \Omega_i^{-1} \ll \tau_1 := \tau_A \approx \tau_c \ll \tau_2 := \tau_{rd}, \quad \text{with } \Omega_i^{-1}/\tau_A := O(\delta); \tag{6}$$

(*b*) for the SCR,

$$\tau_0 := \Omega_i^{-1} \ll \tau_1 := \tau_c \ll \tau_2 := \tau_A \ll \tau_3 = \tau_{rd}, \quad \text{with } \Omega_i^{-1}/\tau_A := O(\delta^2). \tag{7}$$

Here τ_A, τ_c and τ_{rd} are the Alfvén, MHD-collision and resistive diffusion times respectively. The MHD collision time scale τ_c, defined in Edenstrasser (1995), is given by the square root of the inverse collision rates, $\tau_c = (\nu_i^{-1}\nu_e^{-1})^{\frac{1}{2}}$, and the resistive diffusion time is given by the well-known standard expression $\tau_{rd} = \mu_0 L^2/\eta$, where η is the resistivity. Furthermore, each MHD time scale is assumed to be situated between the corresponding kinetic ones, and δ is defined as the geometric mean of δ_i and δ_e, i.e. $\delta := (\delta_e \delta_i)^{\frac{1}{2}}$. Analogously to the kinetic theory, the unique time variable t is now expanded into three (four) formally independent time variables t_n for the ICR (SCR) leading in a standardized form to (Edenstrasser 1995)

$$t = \sum_{n=0} \delta^n t_n, \quad \frac{\partial}{\partial t} = \sum_{n=0} \delta^n \frac{\partial}{\partial t_n}. \tag{8}$$

(iv) The Maxwell equations are also brought into dimensionless form by normalizing all physical quantities $\tilde{Q}$ with respect to some characteristic values (Edenstrasser 1995):

$$\left.\begin{aligned} &\mathbf{j} = \frac{\tilde{\mathbf{j}}}{\bar{j}}, \quad \bar{j} = \frac{en_e v_A}{(\Omega_i \tau_A)}, \quad \mathbf{j} = (\Omega_i \tau_A)[(\mathbf{u}n)^A_{i0} - (\mathbf{u}n)^A_{e0}], \\ &(\mathbf{u}n)^A_\alpha := a_\alpha^{-1} \int \mathbf{v} f_\alpha \, d\mathbf{v}, \end{aligned}\right\} \tag{9a}$$

$$q = \frac{\tilde{q}}{\bar{q}}, \quad \bar{q} = e\bar{n}_e, \quad q = n_i - n_e, \quad n_\alpha := \int f_\alpha \, d\mathbf{v}, \tag{9b}$$

where the superscript A indicates normalization with respect to the Alfvén velocity by (9a). Thus the dimensionless Maxwell equations read

$$\left.\begin{aligned} &\operatorname{curl}\mathbf{B} = \mathbf{j} + \Omega_i \tau_A \left(\frac{v_A}{c}\right)^2 \frac{\partial \mathbf{E}}{\partial t}, \quad \operatorname{div}\mathbf{B} = 0, \\ &\operatorname{curl}\mathbf{E} = -\Omega_i \tau_A \frac{\partial \mathbf{B}}{\partial t}, \quad \operatorname{div}\mathbf{E} = \Omega_i \tau_A \left(\frac{c}{v_A}\right)^2 q. \end{aligned}\right\} \tag{9c}$$

Note that the fluid velocities, the current and the electric field are not normalized to unity. Therefore, in the normalized form of the corresponding expansions, the zeroth- and even the higher-order terms are assumed to be either of order unity or zero.

In the expansion of the following step (v) the ratio v_A/c is assumed to be of order δ (δ^2) in the ICR (SCR). This assumption, which influences the expansions of the displacement current and the charge distribution in the higher-order equations, has to be checked and eventually modified for each special case considered.

(v) The particle distribution function f_α is expanded in the smallness parameter δ_α, and both the self-consistent electric and magnetic fields in the smallness parameter δ in the forms

$$f_\alpha = \sum_{n=0} \delta_\alpha^n f_{\alpha n}, \quad \mathbf{E}(\mathbf{B}) = \sum_{n=0} \delta^n \mathbf{E}_n(\mathbf{B}_n). \tag{10}$$

Substitution from (3) and (10) into the dimensionless Fokker–Planck equation (4b) then yields a separate kinetic equation for each order of δ_α.

(vi) After taking the velocity moments for each order of the dimensionless Fokker–Planck equation, we arrive at the corresponding two-fluid transport equations, which, however, still have a species-dependent time and velocity normalization and which still depend on all kinetic time variables.

(vii) Application of the expansions from (8) and (10) to the dimensionless

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Maxwell equations (9*c*) yields for each order of δ a separate set of dimensionless Maxwell equations.

(viii) Since the two-fluid transport equations must be solved consistently with the corresponding Maxwell equations, they have to be brought into a uniquely normalized form following the same lines as in Edenstrasser (1995).

(ix) To avoid in the limit secular behaviour as $t_0(t_1) \to \infty$, certain terms have to vanish. In order to get the transport equations on a slower time scale, one has to perform the time averages over the preceding ones, for which one assumes stationary or static equilibrium states, with (harmonic) fluctuations superimposed. Time-averaging over these fluctuations will then lead to turbulent contributions in the transport equations on slower time scales. In particular, we are thinking here (because of experimental evidence) of ideal MHD fluctuations.

To illustrate this, let us for example consider the case of two time scales t_1 and t_2, so that the physical quantities q behave as

$$q = q_0(t_1, t_2) + \delta \sum_{n=0} q_{1n}(t_2) \exp(2\pi i n t_1).$$

The equilibrium on the time scale t_1 is characterized by $q_0(t_2) + \delta q_{10}(t_2)$. If in the next-order equation there appear terms quadratic in q_1 then time averaging over t_1 will lead to turbulent contributions.

(x) The two-fluid transport equations are then brought into dimensional form, and, with the usual simplifying assumptions (Edenstrasser 1995), one finally arrives at the one-fluid transport equations.

3. Transport equations for an intermediately collisional plasma

For an intermediately collisional plasma (the ICR), it is assumed that the Alfvén time and the MHD-collision time are of the same order of magnitude. This, however, also implies the relations $\nu_e^{-1} \leqslant \tau_A \leqslant \nu_i^{-1}$ and $\omega_e^{-1} \leqslant \tau_A \leqslant \omega_i^{-1}$. The latter is satisfied for most plasmas, and is only violated for very low-β plasmas with $\beta < 4m_e/m_i$ (see equation (30) of Edenstrasser 1995). For the first of these inequalities to be violated, an analogous relation must hold, of the form $\beta < 4m_e/m_i F(T^2/n)$, where $F(T^2/n)$ is smaller than unity in many cases of interest. In contrast to the case of a weakly collisional plasma, there exist only three distinct time scales, since the Alfvén and MHD-collision time scales are combined into a single time scale. The time scales considered are thus the ion-gyration, Alfvén (collision) and resistive diffusion ones. Examples of plasmas satisfying these intermediate collisional requirements are interplanetary plasmas, the earth's ionospheric plasma and plasmas in early fusion devices with temperatures well below 1 keV.

After performing expansions in the smallness parameters δ_α and δ respectively for the particle distribution functions f_α and the electromagnetic quantities, and applying the multiple-time-scale derivative expansion scheme to the dimensionless Fokker–Planck equation

$$\frac{\partial f_\alpha}{\partial t} + \delta_\alpha \mathbf{v} . \nabla f_\alpha + \sigma(q_\alpha)\{a_\alpha \mathbf{E} + \mathbf{v} \times \mathbf{B}\} . \frac{\partial f_\alpha}{\partial \mathbf{v}} = \delta_\alpha \Lambda_\alpha C_\alpha, \tag{11}$$

we obtain separate equations for each order in δ_α:

$$\delta_\alpha^0: \quad \frac{\partial f_{\alpha 0}}{\partial t_{\alpha 0}} + \sigma(q_\alpha)(a_\alpha \mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0).\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = 0, \tag{12a}$$

$$\delta_\alpha^1: \quad \frac{\partial f_{\alpha 0}}{\partial t_{\alpha 1}} + \frac{\partial f_{\alpha 1}}{\partial t_{\alpha 0}} + \mathbf{v}.\nabla f_{\alpha 0} + \sigma(q_\alpha)(a_\alpha \mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0).\frac{\partial f_{\alpha 1}}{\partial \mathbf{v}} + \sigma(q_\alpha)\left(\frac{\delta_\beta}{\delta_\alpha}\right)^{\frac{1}{2}}(a_\alpha \mathbf{E}_1 + \mathbf{v}\times\mathbf{B}_1).\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = \Lambda_\alpha C_{\alpha 0}(f_{\alpha 0}, f_{\beta 0}), \tag{12b}$$

$$\delta_\alpha^2: \quad \frac{\partial f_{\alpha 0}}{\partial t_{\alpha 2}} + \frac{\partial f_{\alpha 1}}{\partial t_{\alpha 1}} + \frac{\partial f_{\alpha 2}}{\partial t_{\alpha 0}} + \mathbf{v}.\nabla f_{\alpha 1} + \sigma(q_\alpha)(a_\alpha \mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0).\frac{\partial f_{\alpha 2}}{\partial \mathbf{v}} + \sigma(q_\alpha)\left(\frac{\delta_\beta}{\delta_\alpha}\right)^{\frac{1}{2}}(a_\alpha \mathbf{E}_1 + \mathbf{v}\times\mathbf{B}_1).\frac{\partial f_{\alpha 1}}{\partial \mathbf{v}} + \sigma(q_\alpha)\frac{\delta_\beta}{\delta_\alpha}(a_\alpha \mathbf{E}_2 + \mathbf{v}\times\mathbf{B}_2).\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = \Lambda_\alpha C_{\alpha 1}(f_{\alpha 0}, f_{\beta 1}).$$

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In this case δ_α is the ratio of the particle Larmor radius or the plasma macroscopic (hydromagnetic) length.

Comparing the dimensionless Fokker–Planck equations (12) with those in the case of a weakly collisional plasma (the WCR), we see that

(i) the zeroth-order equation is the same for both cases;

(ii) the first-order equation (12*b*) differs from that for the WCR in the presence of the zeroth-order collision operator on the right-hand side;

(iii) in contrast to the WCR, the first-order equation (12*b*) describes not only the time evolution of the particle distribution function on the transit time scale, but also that on the collision time scale;

(iv) the second-order equation (12*c*) now describes the time evolution of the particle distribution function on the classical diffusion time scale.

3.1. *The zeroth-order equations (the ion-gyration time scale)*

In solving the zeroth-order equations, we are only interested in the stationary case, i.e. in the solution in the asymptotic limit as $t_{\alpha 0} \to \infty$. Since the zeroth-order Fokker–Planck equation (12*a*) is identical to that of the WCR-case, the stationary solution is expected to be the same as well, and may be defined as follows (Edenstrasser 1995): in the limit as $t_{\alpha 0} \to \infty$,

$$f_{\alpha 0} \to \text{drifted Maxwellian}, \tag{13a}$$

$$\mathbf{E}_0 + \mathbf{u}_0 \times \mathbf{B}_0 = 0, \tag{13b}$$

$$\mathbf{j}_0 \times \mathbf{B}_0 = 0, \tag{13c}$$

where $\mathbf{j}_0$ is defined by

$$\mathbf{j}_0 = \Omega_i \tau_A[(\mathbf{u}n)_{i0}^A - (\mathbf{u}n)_{e0}^A]. \tag{13d}$$

Thus in the limit as $t_{\alpha 0} \to \infty$, one obtains the force-free equilibria and Ohm's law of ideal MHD. It may be remarked that (12*a*) also allows other stationary solutions. However, since having the zeroth-order distribution function $f_{\alpha 0}$

given by a drifted Maxwellian is to crucial in solving the hierarchy of equations, we may postulate it a priori. That the leading-order distribution function indeed has to be a drifted Maxwellian is shown in the appendix.

3.2. *The first-order equations (the Alfvén or collision time scale)*

Starting with the first-order Fokker–Planck equation (12*b*), and following the derivation scheme outlined in §2, we first obtain the following dimensional two-fluid equations:

$$\frac{\partial \rho_{\alpha 0}}{\partial t} + \nabla \cdot (\rho_{\alpha 0}\, \mathbf{u}_{\alpha 0}) = 0, \tag{14a}$$

$$\rho_{\alpha 0}\frac{\partial}{\partial t}\mathbf{u}_{\alpha 0} + \rho_{\alpha 0}(\mathbf{u}_{\alpha 0}\cdot\nabla)\,\mathbf{u}_{\alpha 0} + \nabla p_{\alpha 0} - q_\alpha\left\{\delta_\alpha\left[n_{\alpha 1}\,\mathbf{E}_0 + \frac{1}{c}(n\mathbf{u})_{\alpha 1}\times\mathbf{B}_0\right] + n_{\alpha 0}\,\delta\left[\mathbf{E}_1 + \frac{1}{c}(n\mathbf{u})_{\alpha 0}\times\mathbf{B}_1\right]\right\} = \mathbf{R}_{\alpha 0}, \tag{14b}$$

$$\frac{\partial}{\partial t}[(\tfrac{3}{2}p_{\alpha 0} + \tfrac{1}{2}\rho_{\alpha 0}\,u_{\alpha 0}^2) + \nabla(\tfrac{5}{2}p_{\alpha 0} + \tfrac{1}{2}\rho_{\alpha 0}\,u_{\alpha 0}^2)\,\mathbf{u}_{\alpha 0}] = q_\alpha[(n\mathbf{u})_{\alpha 0}\cdot\delta\mathbf{E}_1 + \delta_\alpha(n\mathbf{u})_{\alpha 1}\cdot\mathbf{E}_0] + E_{c\alpha 0}. \tag{14c}$$

By combining (14*b*) and (14*c*), under the assumptions that the specific heat ratio $\gamma = \frac{5}{3}$ and $\mathbf{u}_{e0}\times\mathbf{B}_0 \approx \mathbf{u}_{i0}\times\mathbf{B}_0 \approx \mathbf{u}_0\times\mathbf{B}_0$, we obtain the transport equation for the internal energy and the heat balance equation in the equivalent forms

$$\frac{d}{dt}(p_{i0}\,\rho_0^{-\gamma}) = \frac{\partial}{\partial t}(p_{i0}\,\rho_0^{-\gamma}) + \mathbf{u}_0\cdot\nabla(p_{i0}\,\rho_0^{-\gamma}) = \tfrac{2}{3}\rho_0^{-\gamma}\,Q_{i0}, \tag{14d}$$

$$\frac{d}{dt}(p_{e0}\,\rho_0^{-\gamma}) = \frac{\partial}{\partial t}(p_{e0}\,\rho_0^{-\gamma}) + \mathbf{u}_0\cdot\nabla(p_{e0}\,\rho_0^{-\gamma}) = \tfrac{2}{3}\rho_0^{-\gamma}\,Q_{e0} + \frac{1}{en_0}\mathbf{j}_0\cdot\nabla(p_{e0}\,\rho_0^{-\gamma}), \tag{14e}$$

where the frictional force $\mathbf{R}_{\alpha 0}$, the collisional exchange $E_{c\alpha 0}$ and the particle heating energy $Q_{\alpha 0}$ are defined by

$$\mathbf{R}_{\alpha 0} = \delta_\alpha\, m_\alpha\, \bar{n}_\alpha\, \Lambda_\alpha\, \Omega_\alpha\, v_{\text{th}}^\alpha \langle\mathbf{v}\rangle_\alpha = \rho_{e0}\,\nu_{e0}(\mathbf{u}_{\beta 0} - \mathbf{u}_{\alpha 0}) = -\mathbf{R}_{\beta 0}, \tag{15}$$

$$E_{c\alpha 0} = \frac{3}{m_i}\rho_{e0}\,\nu_{e0}(T_{\beta 0} - T_{\alpha 0}) + \mathbf{u}_{i0}\cdot\mathbf{R}_{\alpha 0}, \tag{16}$$

$$Q_{\alpha 0} = E_{c\alpha 0} - \mathbf{u}_{\alpha 0}\cdot\mathbf{R}_{\alpha 0}, \tag{17}$$

where ρ_{e0} is the zeroth-order mass density of the electron fluid and ν_{e0} is the zeroth-order collision rate of the electrons. In comparison with the two-fluid transport equations for the weakly collisional case, it turns out that (14) differ from those obtained on the Alfvén time scale on account of the collision terms on the right-hand side as well as from those for the MHD-collision time scale because of the absence of the turbulent terms and of the first-order heat flux vector and pressure tensor. We further infer that in the case of an intermediately collisional plasma one does not in general arrive at the two-fluid equations of ideal MHD. This would be true, apart from the heat balance equation for the electron fluid, if one assumes a slowly flowing plasma with $|\mathbf{u}_i - \mathbf{u}_e| \ll v_A$ and equal zeroth-order temperature profiles of electrons and ions. If one furthermore assumes a static zeroth-order equilibrium then one also arrives at the heat balance equation of ideal MHD for the electron fluid.

With the help of the usual simplifying assumptions (i.e. $m_e/m_i \ll 1$, $(\rho\mathbf{u})_n \approx (\delta_i/\delta_e)^{\frac{1}{2}n}(\rho\mathbf{u})_{in}$, and $\mathbf{u}_n \approx (\delta_i/\delta_e)^{\frac{1}{2}n}\mathbf{u}_{in}$), we finally arrive at the one-fluid transport equations, where t is related to the Alfvén (collision) time scale:

$$\frac{\partial\rho_0}{\partial t}+\nabla.(\rho_0\mathbf{u}_0)=0, \tag{18a}$$

$$\rho_0\frac{\partial}{\partial t}\mathbf{u}_0+\rho_0(\mathbf{u}_0.\nabla)\mathbf{u}_0+\nabla p_0-\delta\frac{1}{c}(\mathbf{j}_1\times\mathbf{B}_0+\mathbf{j}_0\times\mathbf{B}_1)=0, \tag{18b}$$

$$\frac{\partial}{\partial t}(p_0\rho_0^{-\gamma})+\mathbf{u}_0.\nabla(p_0\rho_0^{-\gamma})=-\tfrac{2}{3}\rho_0^{-\gamma}\mathbf{R}_{e0}.(\mathbf{u}_{e0}-\mathbf{u}_{i0})+\frac{1}{en_0}\mathbf{j}_0.\nabla(p_{e0}\rho_0^{-\gamma}). \tag{18c}$$

In the light of (18), it turns out the introduction of moderate collisions leads to violation of plasma adiabaticity on the Alfvén time scale, this violation being essentially due to the work done by the frictional forces. Note that the fluid velocities are not normalized to unity, but with respect to the Alfvén velocity, so that in the slowly flowing case the zeroth-order fluid velocities have to vanish. Similarly, it can be shown that the first-order Maxwell's equations are written in the following dimensionless form:

$$\operatorname{curl}\mathbf{B}_1=\mathbf{j}_1, \tag{19a}$$

$$\operatorname{div}\mathbf{B}_1=0, \tag{19b}$$

$$\operatorname{curl}\mathbf{E}_0=-\Omega_i\tau_A\delta\frac{\partial\mathbf{B}_0}{\partial t_1}, \tag{19c}$$

$$\mathbf{j}_1=\Omega_i\tau_A\left[\left(\frac{\delta_i}{\delta_e}\right)^{\frac{1}{2}}(\mathbf{u}n)^A_{i1}-\left(\frac{\delta_e}{\delta_i}\right)^{\frac{1}{2}}(\mathbf{u}n)^A_{e1}\right], \tag{19d}$$

$$q_1=\left(\frac{\delta_i}{\delta_e}\right)^{\frac{1}{2}}n_{i1}-\left(\frac{\delta_e}{\delta_i}\right)^{\frac{1}{2}}n_{e1}=0. \tag{19e}$$

For the normalizations used and for the definition of $(\mathbf{u}n)^A_{\alpha1}$, the reader is referred to Edenstrasser (1995). Equations (19) are similar to those of ideal MHD. Thus it can be concluded that the time evolution of the electromagnetic fields is still ideal in nature on the Alfvén (collision) time scale for the ICR. Furthermore, it can be shown that the first-order Ohm's law can be written in the following dimensional form:

$$\mathbf{E}_1=-\frac{1}{c}(\mathbf{u}_0\times\mathbf{B}_1+\mathbf{u}_1\times\mathbf{B}_0)+\frac{1}{c}\frac{\mathbf{j}_0\times\mathbf{B}_1+\mathbf{j}_1\times\mathbf{B}_0}{en_{e0}}+\frac{1}{\delta}\frac{\mathbf{R}_{e0}-\nabla p_{e0}}{en_{e0}}. \tag{20}$$

Again the first-order Ohm's law is different from that obtained for the WCR on account of the zeroth-order electron frictional force $\mathbf{R}_{e0}$. The criterion whether or not the Hall term can be neglected is given by (27) in Edenstrasser (1995). It should be remarked that the first-order Ohm's law (20) is only needed to eliminate the first-order electric field in the two-fluid equations. In (19c) there enters only the zeroth-order electric field subject to the IMHD Ohm's law.

3.3. *The second-order equations (the resistive diffusion time scale)*

Similarly, by following the general derivation scheme outlined in §2, we arrive at the following second-order two-fluid transport equations in dimensional [illegible]

(22c)

With the usual simplifying assumptions (i.e. $m_e/m_i \ll 1$, $(\rho \mathbf{u})_n \approx (\delta_i/\delta_e)^{\frac{1}{2}n} (\rho \mathbf{u})_{in}$ and $\mathbf{u}_n \approx (\delta_i/\delta_e)^{\frac{1}{2}n}\, \mathbf{u}_{in}$), we finally end up with the one-fluid transport equations, where t is related to the resistive diffusion time:

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \mathbf{u}) = 0, \tag{23a}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} . \nabla)\, \mathbf{u} + \nabla . \mathsf{P} - \frac{1}{c} \mathbf{j} \times \mathbf{B} = 0, \tag{23b}$$

$$\frac{d}{dt}(p_e \rho^{-\gamma}) = \tfrac{2}{3} \rho^{-\gamma} \left[Q_e - \nabla . \mathbf{q}_e - (\mathbf{\Pi} . \nabla) \left(\mathbf{u} - \frac{\mathbf{j}}{en} \right) \right] + \frac{\mathbf{j}}{en} . \nabla (p_e \rho^{-\gamma}), \tag{23c}$$

$$\frac{d}{dt}(p_i \rho^{-\gamma}) = \tfrac{2}{3} \rho^{-\gamma} [Q_i - \nabla . \mathbf{q}_i - (\mathbf{\Pi} . \nabla)\, \mathbf{u}]. \tag{23d}$$

Thus the known plasma transport equations for the resistive diffusion time scale (cf. e.g. Freidberg 1987) are obtained, with additional turbulent terms arising from the time averaging over the preceding time scales. Furthermore, note that the higher-order moments $\mathbf{\Pi}_\alpha$, $\mathbf{R}_\alpha$, $\mathbf{q}_\alpha$, and $E_{c\alpha}$ are clearly defined in terms of the zeroth- and first-order distribution functions $f_{\alpha 0}$ and $f_{\alpha 1}$. If, analogously to $f_{\alpha 0}$, $f_{\alpha 1}$ can also be expressed in terms of the fluid variables, as expected, then the higher-order moments will also be functions of the fluid variables, like the zeroth-order frictional forces and the zeroth-order collisional energy exchange energy, $\mathbf{R}_{\alpha 0}$ and $E_{c\alpha 0}$, given in (15)–(17). Similarly, one can show that the second-order Maxwell equations can be written in the dimensionless form

$$\operatorname{curl} \mathbf{B}_2 = \mathbf{j}_2, \tag{24a}$$

$$\operatorname{div} \mathbf{B}_2 = 0, \tag{24b}$$

$$\operatorname{curl} \mathbf{E}_1 = -\Omega_i \tau_A \delta \left(\frac{\partial \mathbf{B}_0}{\partial t_2} + \frac{\partial \mathbf{B}_1}{\partial t_1} \right), \tag{24c}$$

$$\mathbf{j}_2 = \Omega_i \tau_A \left[\frac{\delta_i}{\delta_e} (\mathbf{u}n)^A_{i2} - \frac{\delta_e}{\delta_i} (\mathbf{u}n)^A_{e2} \right], \tag{24d}$$

$$q_2 = \frac{\delta_i}{\delta_e} n_{i2} - \frac{\delta_e}{\delta_i} n_{e2} = 0. \tag{24e}$$

For the normalization used and for the definition of $(\mathbf{u}n)^A_{\alpha 2}$, the reader is again referred to Edenstrasser (1995). Furthermore, it may be shown that the second-order Ohm's law can be written in the dimensional form

$$\mathbf{E}_2 = -\frac{1}{c} (\mathbf{u}_0 \times \mathbf{B}_2 + \langle \mathbf{u}_1 \times \mathbf{B}_1 \rangle + \mathbf{u}_2 \times \mathbf{B}_0) + \frac{1}{c} \frac{\mathbf{j}_0 \times \mathbf{B}_2 + \langle \mathbf{j}_1 \times \mathbf{B}_1 \rangle + \mathbf{j}_2 \times \mathbf{B}_0}{en_{e0}}$$
$$- \frac{n_{e1}}{n_{e0}} \frac{1}{\delta_i} \frac{\mathbf{R}_{e0} - \nabla p_{e0}}{en_{e0}} + \frac{1}{\delta_i} \frac{\mathbf{R}_{e1} - \nabla . \mathsf{P}_{e1}}{en_{e0}} - \frac{1}{c} \left\langle \frac{n_{e1}}{n_{e0}} \frac{1}{\delta_i} \frac{\mathbf{j}_0 \times \mathbf{B}_1 + \mathbf{j}_1 \times \mathbf{B}_0}{en_{e0}} \right\rangle. \tag{25}$$

Again the second-order Ohm's law is different from that obtained for the WCR on account of the first-order electron frictional force $\mathbf{R}_{e1}$. Note that this Ohm's law also includes several turbulent contributions arising from the time averaging of the MHD fluctuation spectra over the preceding time scales.

4. The transport equations for a high-collisional plasma

In this regime just as for a weakly collisional plasma there exist four different time scales, although here the Alfvén and the MHD-collision time scales are interchanged. For a strongly collisional plasma (the SCR), it is assumed that the MHD-collision time is much shorter than the Alfvén time (i.e. $\tau_c \ll \tau_a$). This assumption, however, may only be violated for low-β plasmas satisfying the relation,

$$\beta \ll 4\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \frac{25{\cdot}8\pi^{\frac{1}{2}}\epsilon_0^2 T^4}{Z^5 e^4 \ln \Lambda l^2 n^2}. \tag{26}$$

Examples of plasmas satisfying the strong-collision requirement are inter-galactic and interstellar plasmas.

Expanding the particle distribution functions f_α and the electric (magnetic) field in the smallness parameters δ_α and δ respectively, together with the application of the multiple-time-scale derivative expansion scheme to the dimensionless Fokker–Planck equation

$$\frac{\partial f_\alpha}{\partial t} + \delta_\alpha^2 \mathbf{v}.\nabla f_\alpha + \sigma(q_\alpha)(a_\alpha \mathbf{E} + \mathbf{v}\times\mathbf{B}).\frac{\partial f_\alpha}{\partial \mathbf{v}} = \delta_\alpha \Lambda_\alpha C_\alpha, \tag{27}$$

leads to a separate kinetic equation for each order of δ_α:

$$\delta_\alpha^0: \quad \frac{\partial f_{\alpha 0}}{\partial t_{\alpha 0}} + \sigma(q_\alpha)(a_\alpha \mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0).\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = 0, \tag{28a}$$

$$\delta_\alpha^1: \quad \frac{\partial f_{\alpha 0}}{\partial t_{\alpha 1}} + \frac{\partial f_{\alpha 1}}{\partial t_{\alpha 0}} + \sigma(q_\alpha)(a_\alpha \mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0).\frac{\partial f_{\alpha 1}}{\partial \mathbf{v}} + \sigma(q_\alpha)\left(\frac{\delta_\beta}{\delta_\alpha}\right)^{\frac{1}{2}}(a_\alpha \mathbf{E}_1 + \mathbf{v}\times\mathbf{B}_1).\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = \Lambda_\alpha C_{\alpha 0}(f_{\alpha 0}, f_{\beta 0}), \tag{28b}$$

$$\delta_\alpha^2: \quad \frac{\partial f_{\alpha 0}}{\partial t_{\alpha 2}} + \frac{\partial f_{\alpha 1}}{\partial t_{\alpha 1}} + \frac{\partial f_{\alpha 2}}{\partial t_{\alpha 0}} + \mathbf{v}.\nabla f_{\alpha 0} + \sigma(q_\alpha)(a_\alpha \mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0).\frac{\partial f_{\alpha 2}}{\partial \mathbf{v}} + \sigma(q_\alpha)\left(\frac{\delta_\beta}{\delta_\alpha}\right)^{\frac{1}{2}} \times (a_\alpha \mathbf{E}_1 + \mathbf{v}\times\mathbf{B}_1).\frac{\partial f_{\alpha 1}}{\partial \mathbf{v}} + \sigma(q_\alpha)\frac{\delta_\beta}{\delta_\alpha}(a_\alpha \mathbf{E}_2 + \mathbf{v}\times\mathbf{B}_2).\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = \Lambda_\alpha C_{\alpha 1}(f_{\alpha 0}, f_{\beta 1}), \tag{28c}$$

$$\delta_\alpha^3: \quad \frac{\partial f_{\alpha 0}}{\partial t_{\alpha 3}} + \frac{\partial f_{\alpha 1}}{\partial t_{\alpha 2}} + \frac{\partial f_{\alpha 2}}{\partial t_{\alpha 1}} + \frac{\partial f_{\alpha 3}}{\partial t_{\alpha 0}} + \mathbf{v}.\nabla f_{\alpha 1} + \sigma(q_\alpha)(a_\alpha \mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0).\frac{\partial f_{\alpha 3}}{\partial \mathbf{v}} + \sigma(q_\alpha)\left(\frac{\delta_\beta}{\delta_\alpha}\right)^{\frac{1}{2}}(a_\alpha \mathbf{E}_1 + \mathbf{v}\times\mathbf{B}_1).\frac{\partial f_{\alpha 2}}{\partial \mathbf{v}} + \sigma(q_\alpha)\frac{\delta_\beta}{\delta_\alpha}(a_\alpha \mathbf{E}_2 + \mathbf{v}\times\mathbf{B}_2).\frac{\partial f_{\alpha 1}}{\partial \mathbf{v}} + \left(\frac{\delta_\beta}{\delta_\alpha}\right)^{\frac{3}{2}}(a_\alpha \mathbf{E}_3 + \mathbf{v}\times\mathbf{B}_3).\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = \Lambda_\alpha C_{\alpha 2}. \tag{28d}$$

In this case δ_α is the ratio of the particle mean free path or the plasma macroscopic (hydromagnetic) length. The second-order collision operator $C_{\alpha 2}(f_{\alpha 0}, f_{\beta 2}; f_{\alpha 1}, f_{\beta 1})$ is derived analogously to the first-order one, $C_{\alpha 1}(f_{\alpha 0}, f_{\beta 1})$ in (53) of Edenstrasser (1995).

In comparison with the weakly collisional plasma, the following differences are apparent.

(i) The zeroth-order equation (28*a*) is the same for both cases.

(ii) The first-order equation (28*b*), which describes the time evolution of the distribution function on the collision time scale for a strongly collisional plasma (SCR), differs from the corresponding equation for a weakly collisional plasma on account of the disappearance of the free-flow term $\mathbf{v} . \nabla f_\alpha$. Thus it can be concluded that the spatial variation of the distribution function takes place on a time scale slower than the MHD-collision time scale. Furthermore, the zeroth-order collision operator appears on the right-hand side.

(iii) The second-order equation (28*c*), which describes the time evolution of the particle distribution function on the transit time scale for the SCR is different from the corresponding equation for the WCR on account of the first-order collision operator on the right-hand side.

(iv) The third-order equation (28*d*), which describes the time evolution of the particle distribution function on the classical diffusion time scale, is different from the corresponding one for a weakly collisional plasma on account of the second-order collision operator on the right-hand side.

After performing the velocity moments for each of the dimensionless Fokker–Planck equations (28) and proceeding with the derivation scheme outlined in §2, we obtain a separate set of plasma transport equations for each order of δ.

4.1. *The zeroth-order equations (the ion-gyration timescale)*

In solving the zeroth-order equations, we are only interested in the stationary case, i.e. in the solution for the asymptotic limit as $t_{\alpha 0} \rightarrow \infty$. Since the zeroth-order Fokker–Planck equation (12*a*) is identical to that obtained for the WCR, the stationary solution is also epected to be the same, namely the force-free field equilibria and the ideal MHD Ohm's law (cf. (13)). Thus it can be concluded that the lowest-order gyro-motion is unperturbed by collisions, provided that the particle collision frequency ν_α is much smaller than the gyro-frequency Ω_α (Hinton & Hazeltine 1976).

4.2. *The first-order transport equations (the MHD-Collision time scale)*

Starting with the first-order Fokker–Planck equation (28*b*), and following the general derivation scheme, we first obtain the following dimensional two-fluid equations:

$$\frac{\partial \rho_{\alpha 0}}{\partial t} = 0, \tag{29a}$$

$$\rho_{\alpha 0} \frac{\partial}{\partial t} \mathbf{u}_{\alpha 0} - q_\alpha \left\{ \delta_\alpha \left[n_{\alpha 1} \mathbf{E}_0 + \frac{1}{c} (n\mathbf{u})_{\alpha 1} \times \mathbf{B}_0 \right] + n_{\alpha 0} \delta \left[\mathbf{E}_1 + \frac{1}{c} (n\mathbf{u})_{\alpha 0} \times \mathbf{B}_1 \right] \right\} = \mathbf{R}_{\alpha 0},$$

(29b)

$$\frac{\partial}{\partial t}\left(\tfrac{3}{2}p_{\alpha 0}+\tfrac{1}{2}\rho_{\alpha 0}u_{\alpha 0}^{2}\right)=q_{\alpha}[(n\mathbf{u})_{\alpha 0}\cdot\delta\mathbf{E}_{1}+\delta_{\alpha}(n\mathbf{u})_{\alpha 1}\cdot\mathbf{E}_{0}]+E_{c\alpha 0}. \tag{29c}$$

By combining (29b) and (29c), and assuming that the specific heat ratio $\gamma=\frac{5}{3}$ and $\mathbf{u}_{e0}\times\mathbf{B}_{0}\approx\mathbf{u}_{i0}\times\mathbf{B}_{0}\approx\mathbf{u}_{0}\times\mathbf{B}_{0}$, we obtain the transport equation for the internal energy and the heat balance equation in the equivalent forms

$$\frac{\partial}{\partial t}(p_{i0}\rho_{0}^{-1})=\tfrac{2}{3}\rho_{0}^{-\gamma}Q_{i0}, \tag{29d}$$

$$\frac{\partial}{t}(p_{e0}\rho_{0}^{-1})=\tfrac{2}{3}\rho_{0}^{-\gamma}Q_{e0}, \tag{29e}$$

where the frictional force $\mathbf{R}_{\alpha 0}$, the collisional energy exchange $E_{c\alpha 0}$, and the particle heating term $Q_{\alpha 0}$ are given by (15), (16) and (17) respectively. It is worth noting that, in contrast to the WCR, the above two-fluid equations do not contain turbulent contributions. Furthermore, because of the absence of the convective terms, one concludes that the space dependence of the plasma parameters n_{α}, $\mathbf{u}_{\alpha}$, and T_{α} is governed by slower evolution processes.

With the usual simplifying assumptions (i.e. $m_e/m_i\ll 1$, $(\rho\mathbf{u})_n\approx(\delta_i/\delta_e)^{\frac{1}{2}n}(\rho\mathbf{u})_{in}$, and $\mathbf{u}_n\approx(\delta_i/\delta_e)^{\frac{1}{2}n}\mathbf{u}_{in}$), we finally arrive at the one-fluid transport equations, where t is related to the MHD-collision time:

$$\frac{\partial\rho_0}{\partial t}=0, \tag{30a}$$

$$\rho_0\frac{\partial}{\partial t}\mathbf{u}_0-\delta\frac{1}{c}(\mathbf{j}_1\times\mathbf{B}_0+\mathbf{j}_0\times\mathbf{B}_1)=0, \tag{30b}$$

$$\frac{\partial}{\partial t}(p_0\rho_0^{-\gamma})=-\tfrac{2}{3}\rho_0^{-\gamma}\mathbf{R}_{e0}\cdot(\mathbf{u}_{e0}-\mathbf{u}_{i0}). \tag{30c}$$

Thus, on the MHD collision time scale, we have, apart from the heat balance equation, arrived at the ideal MHD equations. For a slowly flowing plasma with $u_{\alpha 0}\ll v_A$, the right-hand side of (30c) vanishes. Similarly, one can show that the first-order Maxwell equations can be written in the following dimensionless form:

$$\operatorname{curl}\mathbf{B}_1=\mathbf{j}_1, \tag{31a}$$

$$\operatorname{div}\mathbf{B}_1=0, \tag{31b}$$

$$\frac{\partial\mathbf{B}_0}{\partial t_1}=0, \tag{31c}$$

$$\mathbf{j}_1=\Omega_i\tau_A\left[\left(\frac{\delta_i}{\delta_e}\right)^{\frac{1}{2}}(\mathbf{u}n)_{i1}^{A}-\left(\frac{\delta_e}{\delta_i}\right)^{\frac{1}{2}}(\mathbf{u}n)_{e1}^{A}\right], \tag{31d}$$

$$q_1=\left(\frac{\delta_i}{\delta_e}\right)^{\frac{1}{2}}n_{i1}-\left(\frac{\delta_e}{\delta_i}\right)^{\frac{1}{2}}n_{e1}=0. \tag{31e}$$

Moreover, the first-order Ohm's law can be written in the following dimensional form:

$$\mathbf{E}_1 = -\frac{1}{c}(\mathbf{u}_0 \times \mathbf{B}_1 + \mathbf{u}_1 \times \mathbf{B}_0) + \frac{1}{\delta}\frac{\mathbf{R}_{e0}}{en_{e0}} + \frac{1}{c}\frac{\mathbf{J}_0 \times \mathbf{B}_1 + \mathbf{J}_1 \times \mathbf{B}_0}{en_{e0}}. \quad (32)$$

Again the first-order Ohm's law is different from that obtained for the SCR on account of the zeroth-order electron frictional force $\mathbf{R}_{e0}$ as well as the absence of the zeroth-order electron pressure gradient ∇p_{e0}.

4.3. *The second-order transport equations (the Alfvén time scale)*

Following the general scheme of derivation outlined in §2, we analogously arrive at the following second-order two-fluid transport equations in dimensional form:

$$\frac{\partial \rho_{\alpha 0}}{\partial t} + \nabla . (\rho \mathbf{u})_{\alpha 0} = 0, \quad (33a)$$

$$\frac{\partial}{\partial t}(\rho \mathbf{u})_{\alpha 0} + \nabla . (\rho \mathbf{u} \otimes \mathbf{u})_{\alpha 0} + \nabla p_{\alpha 0} - q_\alpha \delta_\alpha^2 \left\{ \left[n_{\alpha 2} \mathbf{E}_0 + \frac{1}{c}\langle (n\mathbf{u})_{\alpha 2} \rangle \times \mathbf{B}_0 \right] + \left(\frac{\delta_\beta}{\delta_\alpha}\right)^{\frac{1}{2}} \left\langle n_{\alpha 1} \mathbf{E}_1 + \frac{1}{c}(n\mathbf{u})_{\alpha 1} \times \mathbf{B}_1 \right\rangle + \frac{\delta_\beta}{\delta_\alpha}\left[n_{\alpha 0} \mathbf{E}_2 + \frac{1}{c}(n\mathbf{u})_{\alpha 0} \times \mathbf{B}_2 \right] \right\} = \delta_\alpha \mathbf{R}_{\alpha 1}, \quad (33b)$$

$$\frac{\partial}{\partial t}(\tfrac{3}{2}p + \tfrac{1}{2}\rho u^2)_{\alpha 0} + \nabla . [(\tfrac{5}{2}p + \tfrac{1}{2}\rho u^2)\mathbf{u}]_{\alpha 0} - q_\alpha \delta_\alpha^2 \left[\langle (n\mathbf{u})_{\alpha 2} \rangle . \mathbf{E}_0 + \left(\frac{\delta_\beta}{\delta_\alpha}\right)^{\frac{1}{2}} \langle (n\mathbf{u})_{\alpha 1} . \mathbf{E}_1 \rangle + \frac{\delta_\beta}{\delta_\alpha}(n\mathbf{u})_{\alpha 0} . \mathbf{E}_2 \right] = \delta_\alpha E_{c\alpha 1}, \quad (33c)$$

where t is now related to the Alfvén time. Furthermore, the first-order frictional fore $\mathbf{R}_{\alpha 1}$ and the first-order collisional energy exchange $E_{c\alpha 1}$ are given by (21f) and (21g) respectively. Note that (33) contain non-vanishing time-averaged quadratic terms, which are essentially turbulent contributions. By adding the zeroth- and first-order equations to (33) and by taking into account our expansion scheme of the physical quantities up to the second order, we formally obtain the following two-fluid transport equations:

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla . (\rho \mathbf{u})_\alpha = 0, \quad (34a)$$

$$\rho_\alpha \frac{\partial \mathbf{u}_\alpha}{\partial t} + \rho_\alpha (\mathbf{u}_\alpha . \nabla)\mathbf{u}_\alpha + \nabla p_\alpha - q_\alpha \left[n_\alpha \mathbf{E} + \frac{1}{c}(n\mathbf{u})_\alpha \times \mathbf{B} \right] = \mathbf{R}_\alpha, \quad (34b)$$

$$\frac{\partial}{\partial t}(\tfrac{3}{2}p_\alpha + \tfrac{1}{2}\rho u_\alpha^2) + \nabla . [(\tfrac{5}{2}p_\alpha + \tfrac{1}{2}\rho u_\alpha^2)\mathbf{u}_\alpha] - q_\alpha (n\mathbf{u})_\alpha . \mathbf{E} = E_{c\alpha}. \quad (34c)$$

In comparison with the two-fluid transport equations for the weakly collisional case, it turns out that (33) differ from those obtained on the Alfvén time scale on account of the first-order collision terms on the right-hand side and the turbulent terms, as well as from those obtained for the MHD-collision time scale on account of the first-order collision terms and the absene of the first-order pressure tensor and the first-order heat flux. It can further be inferred that in the case of a strongly collisional plasma one does not in general arrive at the two-fluid equations of ideal MHD.