Fayoum University
Faculty of Engineering
Dept. of Electrical Engineering
Second Year

19 /06/2011
Mathematics
Second term of 2010/2011
Time : 3 hours

Model answer for the following **SIX** questions:

<u>Question 1</u> [15 points]

i. Draw the curve $|Z-1| = |Z-i|$ in the complex plan.

ii. Find all solutions of the followings:

a) $\operatorname{Sec}(Z) = 2i$.

b) $Z^8 = 256$.

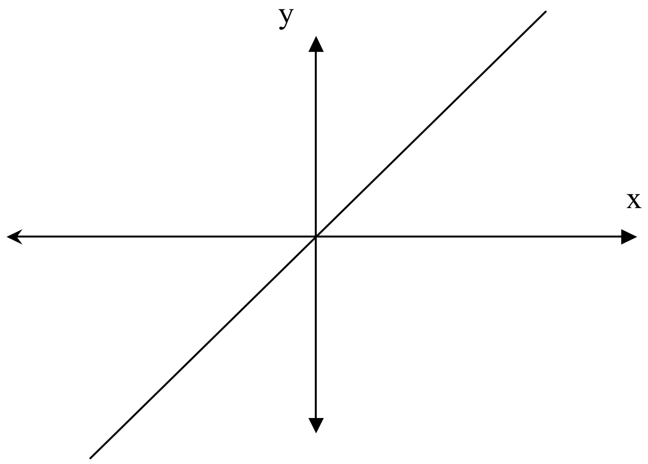
iii. If $f(Z) = \dfrac{Z^2}{|Z|}$, $Z \neq 0$, show that $f(Z)$ isn't differentiable function on C.

iv. If $Z = -1 + i$, evaluate each of $\operatorname{Arg}(Z)$, $\operatorname{Arg}(\overline{Z})$ and $\operatorname{Arg}\left(\dfrac{1}{Z}\right)$.

<u>Solution:</u>

i.

$$\because |Z-1| = |Z-i|$$
$$\therefore |x+iy-1| = |x+iy-i|$$
$$\therefore |(x-1)+iy| = |x+i(y-1)|$$
$$\therefore \sqrt{(x-1)^2+y^2} = \sqrt{x^2+(y-1)^2}$$
$$\therefore x^2-2x+y^2+1 = x^2-2y+y^2+1$$
$$\therefore y = x$$



ii.

a)

$$\because \operatorname{Sec}(Z) = 2i$$

$$\therefore \frac{1}{\operatorname{Cos}(Z)} = 2i$$

$$\therefore \frac{2}{e^{iZ} + e^{-iZ}} = 2i$$

$$\therefore e^{iZ} + e^{-iZ} = -i$$

$$\therefore e^{2iZ} + ie^{iZ} + 1 = 0$$

$$\therefore e^{iZ} = \frac{-i}{2} \pm \frac{\sqrt{-1-4}}{2} = i\left(\frac{-1 \pm \sqrt{5}}{2}\right) = e^{\ln\left(\frac{-1\pm\sqrt{5}}{2}\right) + i\left(\frac{\pi}{2} + 2n\pi\right)}$$

$$\therefore Z = \left(\frac{\pi}{2} + 2n\pi\right) - i\ln\left(\frac{-1 \pm \sqrt{5}}{2}\right) \;;\; \text{where } n = 0, \pm 1, \pm 2, \ldots\ldots$$

b)

$$\because Z^8 = 256$$

$$\therefore Z^8 = 256\, e^{\pm i2n\pi}$$

$$\therefore Z = \sqrt[8]{256}\;\; e^{\frac{\pm i2n\pi}{8}}$$

$$\therefore Z = 2e^{i\frac{n\pi}{4}} ; \quad n = 0,1,2,3,4,5,6,7$$

iii.

$$\because f(Z) = \frac{Z^2}{|Z|} \;,\; Z \neq 0$$

Let $Z = r\, e^{i\theta}$

$$\therefore f(Z) = \frac{r^2 e^{2i\theta}}{r} = re^{2i\theta} = r\left(\operatorname{Cos}(2\theta) + i\operatorname{Sin}(2\theta)\right) = u + iv$$

$$\therefore u = r\operatorname{Cos}(2\theta) \;; \quad v = r\operatorname{Sin}(2\theta)$$

Apply C. – R. equations in polar form $u_r = \frac{v_\theta}{r}$ and $v_r = -\frac{u_\theta}{r}$

$\because u_r = \text{Cos}(2\theta)$ and $\because v_r = \text{Sin}(2\theta)$

$\because u_\theta = -2r\ \text{Sin}(2\theta)$ and $\because v_\theta = 2r\ \text{Cos}(2\theta)$

$\therefore u_r \neq \frac{v_\theta}{r}$ and $v_r \neq -\frac{u_\theta}{r}$

$\therefore f(Z)$ isn't differentiable function

iv.

$\because Z = -1 + i$

$\therefore \text{Arg}(Z) = 135^\circ$

$\because \overline{Z} = -1 - i$

$\therefore \text{Arg}(\overline{Z}) = -135^\circ$ or 225°

$\because \frac{1}{Z} = \frac{1}{-1+i} = \frac{1}{-1+i}\,\frac{-1-i}{-1-i} = \frac{-1-i}{2}$

$\therefore \text{Arg}\left(\frac{1}{Z}\right) = -135^\circ$ or 225°

<u>Question 2</u> [10 points]

Determine the Laurent series for the function $f(Z) = \frac{2}{(Z^2 - 3Z + 2)(Z - 3)}$ in the annulus $1 < |Z| < 2$.

<u>Solution:</u>

$$\because f(Z) = \frac{2}{(Z^2 - 3Z + 2)(Z - 3)}$$

$$\therefore f(Z)=\frac{2}{(Z-1)(Z-2)(Z-3)}=\frac{1}{(Z-1)}-\frac{2}{(Z-2)}+\frac{1}{(Z-3)}$$

$$\because \frac{1}{(Z-1)}=\frac{1/Z}{(1-1/Z)}=\frac{1}{Z}\sum_{n=0}^{\infty}\left(\frac{1}{Z}\right)^n; \qquad |Z|>1$$

$$\because \frac{2}{(Z-2)}=\frac{-1}{(1-Z/2)}=-\sum_{n=0}^{\infty}\left(\frac{Z}{2}\right)^n; \qquad |Z|<2$$

$$\because \frac{1}{(Z-3)}=\frac{-1/3}{(1-Z/3)}=-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{Z}{3}\right)^n; \qquad |Z|<3$$

$$\therefore f(Z)=\frac{1}{Z}\sum_{n=0}^{\infty}\left(\frac{1}{Z}\right)^n+\sum_{n=0}^{\infty}\left(\frac{Z}{2}\right)^n-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{Z}{3}\right)^n; \quad \text{where} \quad 1<|Z|<2$$

Question 3 [10 points]

Classify and define the type of all singularities of the following functions and find their Residue.

i. $f(z)=\frac{\tan(z)}{z^5}$.

ii. $f(Z)=\frac{Z^2}{(Z^2+1)^2}$.

iii. $f(Z)=\frac{e^{Z^2}}{Z^n}$

Solution:

i.

$$f(z)=\frac{\tan(z)}{z^5}$$

Singular point is $Z=0$

For $g(z)=\tan(z)=\sum_{n=0}^{\infty}a_nZ^n$.

$$\because a_4 = \left.\frac{\tan^{(4)}(z)}{4!}\right|_{Z=0} = 0$$

Its type is pole of order 4

Its Residue = zero

ii.

$$\because f(Z) = \frac{Z^2}{\left(Z^2+1\right)^2}$$

$$\therefore f(Z) = \frac{Z^2}{(Z+i)^2(Z-i)^2}$$

Singular points are $Z = \pm i$

their types are pole of order 2

$$\operatorname{Res}[f(Z); Z = i] = \left.\frac{d}{dz}\left[f(Z)(Z-i)^2\right]\right|_{Z=i} = \frac{-i}{4}$$

$$\operatorname{Res}[f(Z); Z = -i] = \left.\frac{d}{dz}\left[f(Z)(Z+i)^2\right]\right|_{Z=-i} = \frac{i}{4}$$

iii.

$$\because f(Z) = \frac{e^{Z^2}}{Z^n} = \frac{1}{Z^n}\sum_{k=0}^{\infty}\frac{Z^{2k}}{k!} = \sum_{k=0}^{\infty}\frac{Z^{2k-n}}{k!}$$

Singular points is $Z = 0$

Its type is pole of order $\frac{n}{2}$ if n is even; and $\frac{n-1}{2}$ if n is odd

$$\operatorname{Res}[f(Z); Z = i] = \begin{cases} \dfrac{1}{\left(\dfrac{n-1}{2}\right)!} & ; \quad n \text{ is odd} \\ \text{zero} & ; \quad n \text{ is even} \end{cases}$$

Question 4 [20 points]

Evaluate the following integrals:

i. $\frac{1}{2\pi i}\oint_C \frac{Cos^n(Z)}{Z^3} dZ \quad , \; C:|Z| = 1; \;\; n \geq 0$

ii. $\oint_C \frac{Z^7 e^{1/Z}}{1 - Z^7} dZ \quad , \; C:|Z| = 2$

Solution:

i. $\frac{1}{2\pi i}\oint_C \frac{Cos^n(Z)}{Z^3} dZ \quad , \; C:|Z| = 1; \;\; n \geq 0$

$$\because \frac{1}{2\pi i}\oint_C \frac{Cos^n(Z)}{Z^3} dZ = \frac{2\pi i}{2!}\frac{d^2}{dZ^2}\left[\frac{Cos^n(Z)}{2\pi i}\right]\Bigg|_{Z=0}$$

$$= \frac{1}{2}\left[n(n-1)Cos^{n-2}(Z)Sin(Z) - nCos^n(Z)\right]\Big|_{Z=0}$$

$$= \frac{-n}{2}$$

ii. $\oint_C \frac{Z^7 e^{1/Z}}{1 - Z^7} dZ \quad , \; C:|Z| = 2$

for $1 - Z^7 = 0$

$\therefore Z^7 = 1 = e^{2\pi i n}\; ; \; n = 0, \pm 1, \pm 2, ...$

$\therefore Z = e^{\frac{2\pi i n}{7}}\; ; \; n = 0, \pm 1, \pm 2, ...$

$\therefore Z = 1, e^{\frac{\pm 2\pi i}{7}}, e^{\frac{\pm 4\pi i}{7}}, e^{\frac{\pm 6\pi i}{7}}$

$$\oint_C \frac{Z^7 e^{1/Z}}{1-Z^7}\, dZ = 2\pi i\{\lim_{Z\to 1}\left(\frac{Z^7 e^{1/Z}}{\left(Z-e^{\frac{2\pi i}{7}}\right)\left(Z-e^{\frac{-2\pi i}{7}}\right)\left(Z-e^{\frac{4\pi i}{7}}\right)\left(Z-e^{\frac{-4\pi i}{7}}\right)}\right)$$

$$+\lim_{Z\to e^{\frac{2\pi i}{7}}}\left(\frac{Z^7 e^{1/Z}}{(Z-1)\left(Z-e^{\frac{-2\pi i}{7}}\right)\left(Z-e^{\frac{4\pi i}{7}}\right)\left(Z-e^{\frac{-4\pi i}{7}}\right)}\right)+\lim_{Z\to e^{\frac{-2\pi i}{7}}}\left(\frac{Z^7 e^{1/Z}}{\left(Z-e^{\frac{2\pi i}{7}}\right)(Z-1)\left(Z-e^{\frac{4\pi i}{7}}\right)\left(Z-e^{\frac{-4\pi i}{7}}\right)}\right)$$

$$+\lim_{Z\to e^{\frac{4\pi i}{7}}}\left(\frac{Z^7 e^{1/Z}}{\left(Z-e^{\frac{2\pi i}{7}}\right)\left(Z-e^{\frac{-2\pi i}{7}}\right)(Z-1)\left(Z-e^{\frac{-4\pi i}{7}}\right)}\right)+\lim_{Z\to e^{\frac{-4\pi i}{7}}}\left(\frac{Z^7 e^{1/Z}}{\left(Z-e^{\frac{2\pi i}{7}}\right)\left(Z-e^{\frac{-2\pi i}{7}}\right)\left(Z-e^{\frac{4\pi i}{7}}\right)(Z-1)}\right)\}$$

(Continue)

Question 5 [20 points]

Show that:

i. $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta} = 2\pi$

ii. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+b^2}\, dx = \frac{\pi e^{-b}}{b}$ Where $b > 0$.

Solution:

i. $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta} = 2\pi$

$$\because \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}$$

Let $Z = e^{i\theta}$

$$\therefore d\theta = \frac{dZ}{Zi}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = -i\oint_C \frac{dZ/Z}{\sqrt{2} - \left(\frac{z + z^{-1}}{2}\right)} \quad \text{; where C is unit circle cetered at zero}$$

$$= -i\oint_C \frac{dZ}{\sqrt{2} - \left(\frac{z + z^{-1}}{2}\right)} = 2i\oint_C \frac{dZ}{\left(Z - \left(\sqrt{2} + 1\right)\right)\left(Z - \left(\sqrt{2} + 1\right)\right)}$$

$$= 2i * 2\pi i \frac{1}{\left(Z - \left(\sqrt{2} + 1\right)\right)}\Bigg|_{Z = \sqrt{2} - 1} = 2\pi$$

ii. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + b^2} dx = \frac{\pi e^{-b}}{b}$ Where $b > 0$

Consider $\displaystyle\oint_C \frac{e^{iZ}}{Z^2 + b^2} dZ$; where C is the upper half of Z - Plane

$$\oint_C \frac{e^{iZ}}{Z^2 + b^2} dZ = \oint_C \frac{e^{iZ}}{(Z + ib)(Z - ib)} dZ = 2\pi i \frac{ie^{Z}}{(Z + ib)}\Bigg|_{Z = ib}$$

$$= 2\pi i \frac{e^{-b}}{2ib}\Bigg| = \frac{\pi e^{-b}}{b}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + b^2} dx = \frac{\pi e^{-b}}{b}$$

Question 6 [10 points]:

Find the Fourier transform of $f(x)e^{ix}$, where $f(x)$ defined by:

$$f(x)=\begin{cases} e^{2|X|} & ,|x|<1 \\ 0 & ,|x|>1 \end{cases}$$

Solution:

$$F\{f(x)e^{ix}\}=F(w-1)$$

$$\because F(w)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{0} e^{-iwx-2x}dx+\int_{0}^{1} e^{-iwx+2x}dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{1-e^{2+iw}}{2+iw}+\frac{-1+e^{2-iw}}{2-iw}\right]$$

$$\therefore F\{f(x)e^{ix}\}=F(w-1)=\frac{1}{\sqrt{2\pi}}\left[\frac{1-e^{2+i(w-1)}}{2+i(w-1)}+\frac{-1+e^{2-i(w-1)}}{2-i(w-1)}\right]$$