Fayoum University | **Model Answer** | Mathematics2 (B)

Faculty of Engineering | | $1^{st}$ year Industrial Eng. Dept

Final Term Exam | June, 13, 2010 | Time: 3 Hrs.

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## *Answer of* Q1 (8 Marks)

| | Differential Equation | Order | Degree | Linear or non-linear? | Homogeneuos or non-homogeneuos? |
|---|---|---|---|---|---|
| a) | $\frac{dx}{dt} + 2t^3 x - x^2 - t = 0$ | 1 | 1 | non-linear | non-homogeneuos |
| b) | $\left(\frac{d^5 y}{dx^5}\right)^3 + x^2 y\left(\frac{d^3 y}{dx^3}\right)^3 - x = 0$ | 5 | 3 | non-linear | non-homogeneuos |
| c) | $\sqrt[3]{(y'')^2} = \sqrt{1 + y'^2}$ | 2 | 4 | non-linear | non-homogeneuos |
| d) | $y'''' - 4y''' - y - x^3 = 0$ | 4 | 1 | linear | non-homogeneuos |

## *Answer of* Q2 (20 Marks)

| | |
|---|---|
| i. | b |
| ii. | b |
| iii. | c |
| iv. | c |
| v. | d |
| vi. | d |
| vii. | a |
| viii. | a |
| ix. | b |
| x. | b |

**Answer of Q3** **(12 Marks)**

**a)** $\dfrac{dy}{dx}=\dfrac{y(y^2-1)}{x(xy^3+1)} \Rightarrow \dfrac{dx}{dy}=\dfrac{x^2y^3+x}{y(y^2-1)} \Rightarrow \dfrac{dx}{dy}-\dfrac{x}{y(y^2-1)}=\dfrac{x^2y^2}{y^2-1}$

$\Rightarrow \dfrac{dx}{dy}-\dfrac{x}{y(y^2-1)}=\dfrac{x^2y^2}{y^2-1} \Rightarrow$ **This is Bernoulli's D.E. ( *n*=2)**

$let\ x^{-1}=z \Rightarrow \dfrac{dx}{dy}=-x^2\dfrac{dz}{dy} \Rightarrow \ -x^2\dfrac{dz}{dy}-\dfrac{x}{y(y^2-1)}=\dfrac{x^2y^2}{y^2-1}$

$\Rightarrow \dfrac{dz}{dy}+\dfrac{z}{y(y^2-1)}=\dfrac{-y^2}{y^2-1} \Rightarrow$ **This is Linear D.E.** $\Rightarrow \ \mu=e^{\int \frac{dy}{y(y^2-1)}}$

$\Rightarrow$ **This integral can by simplified by partial fractions**

$\mu=e^{\int\left(\frac{A}{y}+\frac{B}{(y-1)}+\frac{C}{(y+1)}\right)dx} = e^{\int\left(\frac{-1}{y}+\frac{1/2}{(y-1)}+\frac{1/2}{(y+1)}\right)dx} \Rightarrow \mu=e^{\ln\frac{\sqrt{y^2-1}}{|y|}}=\dfrac{\sqrt{y^2-1}}{y}$

$\mu=e^{\ln\frac{\sqrt{y^2-1}}{|y|}}=\dfrac{\sqrt{y^2-1}}{y} \Rightarrow \dfrac{\sqrt{y^2-1}}{y}z=\int\dfrac{\sqrt{y^2-1}}{y}\dfrac{(-y^2)}{y^2-1}dy=-\int y(y^2-1)^{-1/2}dy$

$=\dfrac{-1}{2}\dfrac{\sqrt{y^2-1}}{1/2}+c \Rightarrow x^{-1}=z=-y+\dfrac{cy}{\sqrt{y^2-1}} \qquad \Rightarrow \boxed{1/x=\dfrac{cy}{\sqrt{y^2-1}}-y}$

**b)** $(y-x)^2dy-\left[(y-x)^2-1\right]dx=0 \Rightarrow y'=\dfrac{(y-x)^2-1}{(y-x)^2}$.

**This D.E. is reduced to separable using the substitiution** $z=y-x \Rightarrow \ z'=y'-1$

$\Rightarrow z'+1=\dfrac{z^2-1}{z^2} \Rightarrow z'=\dfrac{z^2-1-z^2}{z^2} \Rightarrow dx=z^2dz \Rightarrow \int dx=\int z^2dz$

$x+c=\dfrac{z^3}{3} \Rightarrow x+c=\dfrac{(y-x)^3}{3} \qquad \Rightarrow \boxed{y=x+\sqrt[3]{3(x+c)}}$

**Answer of Q4** **(12 Marks)**

**a)** $y'' - \frac{1}{x\ln x}y' = \ln x \Rightarrow$ **Put** $z = y' \Rightarrow z' - \frac{1}{x\ln x}z = \ln x \Rightarrow$

**(linear F.O F.D D.E)** $\Rightarrow \mu = e^{-\int(1/x\ln|x|)dx} = e^{\ln|(1/\ln|x|)|} = \frac{1}{\ln|x|}$

$\Rightarrow z = \ln|x|\left[\int dx + c_1\right] \Rightarrow z = (x + c_1)\ln|x| \Rightarrow y' = (x + c_1)\ln|x| \Rightarrow$

**using the boundary condition:** $y'(e) = e \Rightarrow c_1 \Rightarrow e = (e + c_1) \Rightarrow c_1 = 0$

$\Rightarrow y = \int x\ln|x|dx$ **using the integration by parts technique, we can write:**

$y = \frac{x^2}{2}\ln|x| - \int \frac{x^2}{2}\frac{1}{x}dx \Rightarrow\Rightarrow y = \frac{x^2}{4}\left(2\ln|x| - 1\right) + c_2$

$\Rightarrow y = \frac{x^2}{2}\ln|x| - \int \frac{x}{2}dx$ **, using the boundary condition:** $y(1) = -1/4$

$\Rightarrow -\frac{1}{4} = 0 - \frac{1}{4} + c_2 \Rightarrow c_2 = 0 \Rightarrow$ $\boxed{y = \frac{x^2}{4}\left(2\ln|x| - 1\right)}$

**b)** $y'''' + 2y''' - 3y'' = 4\sin x \Rightarrow y = y_h + y_p \Rightarrow$

$m^4 + 2m^3 - 3m^2 = 0 \Rightarrow m^2(m^2 + 2m - 3) = 0 \Rightarrow m = 0,0,1,\ -3 \Rightarrow$

$y_1 = 1,\ y_2 = x\ ,\ y_3 = e^x\ ,\ y_4 = e^{-3x} \Rightarrow y_h = c_1 + c_2x + c_3e^x + c_4e^{-3x}$

$y_p = \left.\frac{4}{(D^4 + 2D^3 - 3D^2)}\right|_{D^2=-1} \sin x = \frac{-2}{(D-2)}\sin x = \left.\frac{-2(D+2)}{(D^2 - 4)}\right|_{D^2=-1} \sin x$

$y_p = \frac{2}{5}(\cos x + 2\sin x)$ $\Rightarrow \boxed{y = c_1 + c_2x + c_3e^x + c_4e^{-3x} + \frac{2}{5}(\cos x + 2\sin x)}$

**Answer of Q5** **(12 Marks)**

**a)** $y(t) = \lambda^{-1}(Y(s)) = \lambda^{-1}\left( \dfrac{(s+2)\, e^{-\pi s}}{(s^2+1)(s^2+4)} \right) \Rightarrow$ **(Using Partial fraction )**

$$\frac{(s+2)}{(s^2+1)(s^2+4)} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+4)} \Rightarrow (s+2) = (As+B)\,(s^2+4) + (Cs+D)(s^2+1)$$

$$\Rightarrow A = 1/3,\ \ B = 2/3,\quad C = -1/3,\ \ D = \text{-}2/3$$

$$\Rightarrow y(t) = \frac{1}{3}\lambda^{-1}\left( \frac{s+2}{(s^2+1)} e^{-\pi s} \right) - \frac{1}{3}\lambda^{-1}\left( \frac{s+2}{(s^2+4)} e^{-\pi s} \right)$$

$$y(t) = \frac{1}{3}\lambda^{-1}\left( \frac{s}{(s^2+1)} e^{-\pi s} \right) + \frac{2}{3}\lambda^{-1}\left( \frac{1}{(s^2+1)} e^{-\pi s} \right) - \frac{1}{3}\lambda^{-1}\left( \frac{s}{(s^2+2^2)} e^{-\pi s} \right) - \frac{1}{3}\lambda^{-1}\left( \frac{2}{(s^2+2^2)} e^{-\pi s} \right)$$

**Using the t-shifting property, we can write:**

$$\boxed{y(t) = \frac{u_\pi(t)}{3}\left[\cos(t-\pi) + 2\sin(t-\pi) - \cos 2(t-\pi) - \sin 2(t-\pi)\right]}$$

**b)** $y(t) = \lambda^{-1}(Y(s)) = \lambda^{-1}\left( \dfrac{1}{s^3(s^2-1)} \right) = \lambda^{-1}\left( \dfrac{1/(s^2-1)}{s^3} \right) = \lambda^{-1}\left( \dfrac{1/s^2(s^2-1)}{s} \right)$

**(Using property** $\dfrac{F(s)}{s} = \displaystyle\int_0^t f(u)\, du$ **three times , we can write:**

$$y(t) = \int_0^t \lambda^{-1}\left[ \frac{1}{s^2(s^2-1)} \right] dt = \int_0^t \int_0^t \lambda^{-1}\left[ \frac{1}{s(s^2-1)} \right] dt\, dt$$

$$\Rightarrow y(t) = \int_0^t \int_0^t \int_0^t \lambda^{-1}\left[ \frac{1}{(s^2-1)} \right] dt\, dt\, dt = \int_0^t \left[ \int_0^t \left( \int_0^t \sinh t\, dt \right) dt \right] dt$$

$$\Rightarrow y(t) = \int_0^t \left[ \int_0^t (\cosh t - 1)\, dt \right] dt = \int_0^t \left[ \sinh t - t \right] dt$$

$$\boxed{y(t) = \cosh t - \frac{t^2}{2} - 1}$$

**c)** $\dddot{y} + \dot{y} = e^{t}$ **, where,** $y(0) = \dot{y}(0) = \ddot{y}(0) = 0$

$$\Rightarrow s^{3}Y(s) - s^{2}y(0) - sy'(0) - y''(0) + sY(s) - y(0) = \frac{1}{s-1}$$

$$\Rightarrow Y(s) = \frac{1}{(s-1)(s^{3}+s)} = \frac{1}{s(s-1)(s^{2}+1)}$$ **(Using Partial fraction )**

$$\frac{1}{s(s-1)(s^{2}+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^{2}+1}$$

$$1 = A(s-1)(s^{2}+1) + Bs(s^{2}+1) + (Cs+D)s(s-1)$$

$$\Rightarrow A = -1, \;\; B = 1/2, \;\; C = 1/2, \;\; D = -1/2$$

$$\Rightarrow Y(s) = -\frac{1}{s} + \frac{1/2}{s-1} + \frac{s/2 - 1/2}{s^{2}+1} = -\frac{1}{s} + \frac{1/2}{s-1} + \frac{s/2}{s^{2}+1} - \frac{1/2}{s^{2}+1}$$

$$y(t) = -\lambda^{-1}\left(\frac{1}{s}\right) + \frac{1}{2}\lambda^{-1}\left(\frac{1}{(s-1)}\right) + \frac{1}{2}\lambda^{-1}\left(\frac{s}{(s^{2}+1^{2})}\right) - \frac{1}{2}\lambda^{-1}\left(\frac{1}{(s^{2}+1^{2})}\right)$$

$$\boxed{y(t) = -1 + \frac{1}{2}e^{t} + \frac{1}{2}\cos t - \frac{1}{2}\sin t}$$

**Answer of Q6** **(6 Marks)**

$$f(x) = |x| = \begin{cases} -x & -4 \le x < 0 \\ x & 0 \le x \le 4 \end{cases} \quad \Rightarrow \quad f(x) \text{ is an even function} \Rightarrow b_n = 0$$

$$a_o = \frac{2}{4}\int_0^4 x dx = \frac{1}{2}\frac{x^2}{2}\Big|_0^4 = 4.$$

$$a_n = \frac{2}{4}\int_0^4 x\cos\frac{n\pi x}{4}dx = \frac{1}{2}\left( x\sin\frac{n\pi x}{4}\left(\frac{4}{n\pi}\right)\Big|_0^4 - \int_0^4 \sin\frac{n\pi x}{4}\left(\frac{4}{n\pi}\right)dx \right)$$

$$= \frac{1}{2}\left(\frac{4}{n\pi}\right)^2 \cos\frac{n\pi x}{4}\Big|_0^4 = \frac{8}{n^2\pi^2}(\cos n\pi - 1) \Rightarrow a_n = \begin{cases} \frac{-16}{n^2\pi^2} & n \quad odd \\ 0 & n \quad even \end{cases}$$

$\Rightarrow$ **Put** ***n=2m-1,*** ***m=1,2,3,4,………..***

$$\boxed{f(x) = 2 + \sum_{m=1}^{\infty} \frac{-16}{(2m-1)^2\pi^2}\cos(2m-1)\frac{\pi}{4}x}$$

**Thus, if we put (*x*=0) in the above series we may write:**

$$0 = 2 + \sum_{m=1}^{\infty} \frac{-16}{(2m-1)^2\pi^2}(1) \Rightarrow$$

$$\boxed{\sum_{m=1}^{\infty} \frac{8}{(2m-1)^2\pi^2} = 1}$$

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مع دعواتي بالتفوق ..................................... د.م / كارم محمود