Fayoum University
Faculty of Engineering

Preparatory Year Midterm Exam December,1,2010
Prof. Dr. Samy El Badawy Yehia Time : 90 Min.

الإسم : الفصل : رقم الجلوس :

All the questions are of equal weight except number 5 is half the weight of the others

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ANSWERS

1) Prove by Mathematical Induction that ; The sum of n terms of the series :

$$\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+......is \quad \frac{n(3n+5)}{4(n+1)(n+2)}.$$

Answer :

1) For n=1, LHS= $\frac{1}{1.3}=\frac{1}{3}$ RHS= $\frac{1(3+5)}{4(1+1)(1+2)}=\frac{1}{3}$ then true for n=1.

2) Assume true for n = k. Then

$$\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+.....+\frac{1}{k(k+2)}=\frac{k(3k+5)}{4(k+1)(k+2)}.$$

3) For n = k + 1 ;
LHS =

$$\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+.....+\frac{1}{k(k+2)}+\frac{1}{(k+1)(k+3)}=\frac{k(3k+5)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+3)}=$$

$$=\frac{k(3k+5)(k+3)+4(k+2)}{4(k+1)(k+2)(k+3)}=\frac{3k^3+14k^2+15k+4k+8}{4(k+1)(k+2)(k+3)}=$$

$$=\frac{3k^3+14k^2+19k+8}{4(k+1)(k+2)(k+3)}=\frac{(k+1)(3k^2+11k+8)}{4(k+1)(k+2)(k+3)}=\frac{(k+1)(k+1)(3k+8)}{4(k+1)(k+2)(k+3)}=$$

$$=\frac{(k+1)\left[3(k+1)+5\right]}{4\left[(k+1)+1\right]\left[(k+1)+2\right]}.$$

Which is true for n = k+1.

2) Show that : $\dfrac{(1+x+x^2)(1+x)^2}{1-x+x^2}$ = $1+4x+7x^2+6x^3$ if powers beyond x^3 are neglected.

Answer :

$$\frac{(1+x+x^2)(1+x)^2}{1-x+x^2} = \frac{(1+x+x^2)(1+x)^2(1+x)}{(1-x+x^2)(1+x)} = \frac{(1+x+x^2)(1+x)^3}{1+x^3} =$$

$$= (1+x+x^2)(1+3x+3x^2+x^3)(1+x^3)^{-1} =$$

$$=(1+3x+3x^2+x^3+x+3x^2+3x^3+x^2+3x^3+..........)(1+x^3)^{-1} =$$

$$= (1+4x+7x^2+7x^3+......)(1-x^3+........) = (1+4x+7x^2+7x^3-x^3+.......) =$$

$= 1+4x+7x^2+6x^3 +$ terms of higher powers of x.

Another solution:

$$\frac{(1+x+x^2)(1+x)^2}{1-x+x^2} = (1+x+x^2)(1+2x+x^2)[1-(x-x^2)]^{-1} =$$

$$= (1+2x+x^2+x+2x^2+x^3+x^2+2x^3+x^4)[1+(x-x^2)+(x-x^2)^2+(x-x^2)^3+...]$$

$$= (1+3x+4x^2+3x^3+....)(1+x-x^2+x^2-2x^3+x^4+x^3+....)$$

$$= (1+3x+4x^2+3x^3+....)(1+x-x^3+...) = 1+3x+4x^2+3x^3+x+3x^2+4x^3-x^3 =$$

$$= 1+4x+7x^2+6x^3$$

3) Let f(x) be a polynomial with real coefficients. Let $i = \sqrt{-1}$. Assume that $\alpha + \beta i$ is a root for f(x) , where both α, β are real numbers ($\beta \neq 0$) . Prove that $\alpha - \beta i$ is also a root for f(x).

Answer :

Let $d(x) = (x - \alpha - \beta i)(x - \alpha + \beta i)$. A polynomial of degree 2.

Then by the remainder and factor theorem there is a polynomial q(x) and a polynomial of degree one (ax + b) such that :

$$f(x) = q(x)d(x) + ax + b \, .$$

Since $\alpha + \beta i$ is a root for f(x), then :

$f(\alpha + \beta i) = q(\alpha + \beta i) \;\; d(\alpha + \beta i) \;\; + \;\; a(\alpha + \beta i) + b \;= 0.$

Since ; $d(x) = (x - \alpha - \beta i)(x - \alpha + \beta i)$.

Then

$d(\alpha + \beta i) = 0$

and we get :

$a(\alpha + \beta i) + b = 0.$

which gives $a\beta = 0$ then a = 0 since $\beta \neq 0$ and so b = 0.

Then :

$f(x) = q(x)d(x) = \; q(x)(x - \alpha - \beta i)(x - \alpha + \beta i)$

Which means $\alpha - \beta i$ is also a root for f(x).

4) Prove that :

$$\cos^{-1} x = 2\tan^{-1}\frac{\sqrt{1-x}}{\sqrt{1+x}}$$

Answer :

Let

$y = \tan^{-1}\dfrac{\sqrt{1-x}}{\sqrt{1+x}}$.

Then :

$$\sin y = \frac{\sqrt{1-x}}{\sqrt{2}} \quad \text{and} \quad \cos y = \frac{\sqrt{1+x}}{\sqrt{2}}.$$

So :

$$\cos 2y = \cos^2 y - \sin^2 y = \frac{1+x}{2} - \frac{1-x}{2} = x.$$

and

$2y = \cos^{-1} x$.

Which gives :

$$\cos^{-1} x = 2\tan^{-1}\frac{\sqrt{1-x}}{\sqrt{1+x}}.$$

5) Find $\frac{dy}{dx}$ given that : y = $\sinh^{-1}(\tan x)$, simplify your answer, note that $\sqrt{x^2} = |x|$.

Answer :

$$\frac{dy}{dx} = \frac{1}{\sqrt{\tan^2 x + 1}} \frac{d}{dx} \tan x = \frac{1}{\sqrt{\sec^2 x}} \sec^2 x = \frac{1}{|\sec x|} (|\sec x|)^2 = |\sec x| .$$

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6) Let y = $x^{\sin^{-1} x}$ and z = $\sin^{-1} x$, find $\frac{dy}{dz}$ (in terms of x).

Answer :

First let us find $\frac{dy}{dx}$;

$\ln y = (\sin^{-1} x)(\ln x)$. Differentiate with respect to x, we get :

$$(\frac{1}{y})(\frac{dy}{dx}) = \frac{\sin^{-1} x}{x} + (\ln x)(\frac{1}{\sqrt{1-x^2}}).$$

Then :

$$\frac{dy}{dx} = x^{\sin^{-1} x} \left[\frac{\sin^{-1} x}{x} + (\ln x)(\frac{1}{\sqrt{1-x^2}} \right] .$$

Then :

$$\frac{dy}{dz} = \frac{dy}{dx} . \frac{dx}{dz} = \left[\cos(\sin^{-1} x)\right] . \ x^{\sin^{-1} x} \left[\frac{\sin^{-1} x}{x} + (\ln x)(\frac{1}{\sqrt{1-x^2}} \right]$$

Note that : x = sin z then $\frac{dx}{dz} = \cos z = \cos(\sin^{-1} x)$.