Fayoum University
Faculty of Engineering

Preparatory Year, Mathematics — First Term Final Exam.

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3 Hours

ANSWERS

1) Prove by induction that $x^n - y^n$ is divisible by $x+y$ when n is even.

Answer :

1- For n = 2 . $x^2 - y^2 = (x-y)(x+y)$ which is divisible by $x+y$.

2- Assume true for n = k. Then $x^k - y^k = (x+y)f(x,y)$.

3- For n = k + 2 : $\dfrac{x^{k+2} - y^{k+2}}{x+y} = x^{k+1} - x^k y + y^2 \dfrac{x^k - y^k}{x+y} =$

$= x^{k+1} - x^k y + y^2 f(x,y)$. Which means $x^{k+2} - y^{k+2}$ is divisible by $x+y$.

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2) Approximate the value of $\sqrt{\dfrac{97}{101}}$ by expanding the binomial theorem to four nonzero terms.

Answer :

$$\sqrt{\frac{97}{101}} = \sqrt{\frac{101-4}{101}} = (1-\frac{4}{101})^{1/2} = (1+\frac{-4}{101})^{1/2} = 1 + 1/2(\frac{-4}{101}) + \frac{(\frac{1}{2})(\frac{1}{2}-1)}{2!}(\frac{-4}{101})^2 +$$

$$+\frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(\frac{-4}{101})^3 = 1- 0.0198020-0.0001961-0.0000039 =$$

= 1-0.020002 = 0.979998.

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3) Find $\dfrac{dy}{dx}$ for the following : (don't simplify)

a) $y^2 \cot(5y) - \operatorname{cosec}(xy) = 1$.

b) y = $\sqrt{\sec^{-1}(x^{5)}}$.

c) y = $\left[\cosh^{-1}(xy)\right]^{\ln x}$.

Answer :

a) $(y^2)(-\operatorname{cosec}^2(5y))(5y\dfrac{dy}{dx}) + (2y)(\dfrac{dy}{dx})(\cot(5y)) - (-\operatorname{cosec}(xy)\cot(xy)(x\dfrac{dy}{dx} + y) = 0$

b) $$\frac{dy}{dx} = \frac{\dfrac{5x^4}{x^5\sqrt{x^{10}-1}}}{2\sqrt{\sec^{-1}(x^5)}}$$

c) $(\ln y) = (\ln x)\left[\ln(\cosh^{-1}(xy))\right]$

$$\frac{y'}{y} = (\ln x)\left\{\frac{\dfrac{xy' + y}{\sqrt{(xy)^2 - 1}}}{\cosh^{-1}(xy)}\right\} + \left(\frac{1}{x}\right)[\ln(\cosh^{-1}(xy))]$$

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4) Consider the function $y = f(x) = \dfrac{x}{\sqrt{x^2 - 1}}$. Given that :

$\dfrac{dy}{dx} = \dfrac{-1}{(x^2 - 1)^{3/2}}$ and $\dfrac{d^2y}{dx^2} = \dfrac{3x}{(x^2 - 1)^{5/2}}$. Find :

a) The domain of f(x).
b) The vertical and horizontal asymptotes if they exist.
c) Local maximum and local minimum points if they exists.
d) Points of inflection if they exists.
e) Sketch the graph of f(x).

Answer :

a) The domain of f(x) is $(-\infty, -1) \cup (1, \infty)$ Or x < -1 and x>1.

b) Vertical asymptote : x = 1 and x = -1.

Horizontal asymptote :

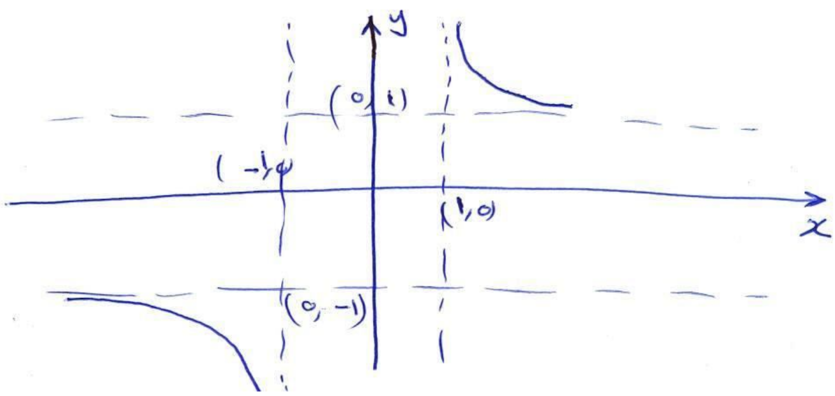
1) To the right : $y = \lim_{x \to \infty} \dfrac{x}{\sqrt{x^2 - 1}} = 1$. So y = 1.

2) To the left : $y = \lim_{x \to -\infty} \dfrac{x}{\sqrt{x^2 - 1}} = -1$. So y = -1.

c) $f'(x) = \dfrac{-1}{(x^2 - 1)^{3/2}}$ So there is no critical points so no local extreme.

d) $f''(x) = \dfrac{3x}{(x^2 - 1)^{5/2}}$ So there is no points of inflection.

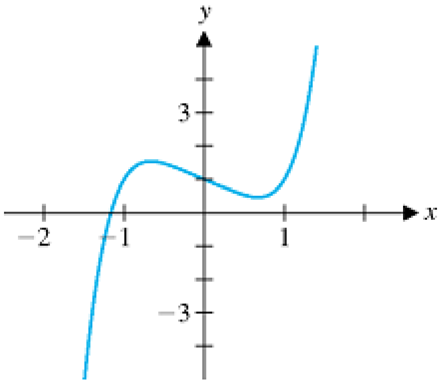
e)



5) Given the graph of the function $y = x^5 - x + 1$. Find the zero correct to three decimals.

$f'(x) = 5x^4 - 1$ and so, Newton's method gives us

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^5 - x_n + 1}{5x_n^4 - 1}, \quad n = 0, 1, 2, \ldots.$$



$y = x^5 - x + 1.$

Using the initial guess $x_0 = -1$, we get

$$x_1 = -1 - \frac{(-1)^5 - (-1) + 1}{5(-1)^4 - 1}$$
$$= -1 - \frac{1}{4} = -\frac{5}{4}.$$

Likewise, from $x_1 = -\frac{5}{4}$, we get the improved approximation

$$x_2 = -\frac{5}{4} - \frac{\left(-\frac{5}{4}\right)^5 - \left(-\frac{5}{4}\right) + 1}{5\left(-\frac{5}{4}\right)^4 - 1}$$
$$= -1.178459394$$

and so on, we find that

$$x_3 = -1.167537384,$$
$$x_4 = -1.167304083$$

6) Find the following limits :

a) $\lim_{x \to 0} (\cot^2 x - \operatorname{cosec}^2 x).$

b) $\lim_{x \to (\pi/2)^-} (1 + \cos x)^{\tan x}.$

Answers :

a) $\lim_{x \to 0} (\cot^2 x - \operatorname{cosec}^2 x) = \lim_{x \to 0} \left(\frac{\cos^2 x}{\sin^2 x} - \frac{1}{\sin^2 x}\right) = \lim_{x \to 0} \left(\frac{\cos^2 x - 1}{\sin^2 x}\right) = -1.$

b) $y = (1 + \cos x)^{\tan x} \Rightarrow \ln y = \tan x \{\ln(1 + \cos x)\}.$

Then

$$\lim_{x \to (\pi/2)^-} (\ln y) = \lim_{x \to (\pi/2)^-} \tan x \{\ln(1 + \cos x)\} = \infty.0 = \lim_{x \to (\pi/2)^-} \frac{\ln(1 + \cos x)}{\cot x} = \frac{0}{0} =$$

$$= \lim_{x \to (\pi/2)^-} \frac{\dfrac{-\sin x}{1+\cos x}}{-\operatorname{cosec}^2 x} = \lim_{x \to (\pi/2)^-} \frac{\sin^3 x}{1+\cos x} = 1.$$

Then $\lim_{x \to (\pi/2)^-} (1+\cos x)^{\tan x} = e.$

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7) If the roots of the equation :

$$x^3 - 7x^2 + \lambda x - 8 = 0$$

form a geometric sequence. Find λ and the roots.

Answer :

The roots are a, ar, ar^2 . then $a^3 r^3 = (-1)^3(-8) = 8.$ Then $ar = 2$ or $a = 2/r.$

The sum of the roots : $a + ar + ar^2 = -(-7) = 7.$ So $\left(2/r\right)(1 + r + r^2) = 7.$ This gives $2r^2 - 5r + 2 = 0.$ Then r = 2 and a = 1. Or r = 1/2 and a = 4. Then the roots are 1 , 2 , 4. So $1 - 7 + \lambda - 8 = 0. And \lambda = 14.$