.

L-FUZZY TOPOGENOUS ORDERS AND L-FUZZY TOPOLOGIES

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Abstract:

In this paper, we introduce the notions of L-fuzzy topoenous orders and investigate some of properties. We investigate the relationships among L-fuzzy topoenous orders, L-fuzzy topologies and L-fuzzy interior operators.

Keywords:

Quantales, L-fuzzy topologies, L-fuzzy topoenous orders, L-fuzzy interior operators

1 Introduction

Šostak [19] introduced a new definition of L-fuzzy topology as the concept of the degree of the openness of a fuzzy set. It is an extension of $[0, 1]$-topology defined by Chang [3]. Höhle and Šostak [8] substituted a lattice L (GL-monoid, cqm-lattice) for the unit interval or the two-point lattice $2 = \{0, 1\}$ in the definitions of [0,1]-(fuzzy) topologies and [0,1]-fuzzy closure spaces in [3,4,6,10,12]. Kim and Min [11] studied L-fuzzy preproximities and L-fuzzy topologies where L is a strictly two-sided, commutative quantale lattice having a strong negation.

In this paper, we introduce the notions of L-fuzzy topoenous orders and investigate some of properties. We investigate the relationships among L-fuzzy topoenous orders, L-fuzzy topologies and L-fuzzy interior operators.

These structures are extensions of $[0, 1]$-(fuzzy) topogenous and $[0, 1]$-(fuzzy) interior operators in [1,2,13-17].

2.1 Preliminaries

Throughout this paper, let X be a nonempty set and $L = (L, \leq, \vee, \wedge, 0, 1)$ a complete lattice where 0 and 1 denote the least and the greatest elements in L. If $a \leq b$ or $b \leq a$ for each $a, b \in L$, then L is called a *chain*. A lattice L is called *order dense* if for each $a, b \in L$ such that $a < b$, there exists $c \in L$ such that $a < c < b$. For each $\alpha \in L$, let $\overline{\alpha}$ denote the constant fuzzy subset of X with value α and $L_0 = L - \{0\}$.

Definition 2.1. [7,8,11]. A complete lattice $(L, \leq, \odot)$ is called a *strictly two-sided, commutative quantale* (scq-lattice , for short) iff it satisfies the following properties

(L1) $(L, \odot)$ is a commutative semigroup.

(L2) $x = x \odot 1$, for each $x \in L$ and 1 is the universal upper bound.

(L3) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} r_i) \odot s = \bigvee_{i\in\Gamma} (r_i \odot s).$$

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Definition 2.2. [7,8,11]. Let $(L, \leq, \odot)$ be a scq-lattice. A mapping $n : L \to L$ is called a *strong negation* , denoted by $n(a) = a^*$, if it satisfies the following conditions:
(N1) $n(n(a)) = a$ for each $a \in L$.
(N2) If $a \leq b$ for each $a, b \in L$, then $n(a) \geq n(b)$.

Remark 2.3.[11]. The following lattices $(L, \leq, \odot, ^*)$ from (1) to (3) are scq-lattices with a strong negation *.
(1) Every completely distributive lattice $(L, \leq, \wedge, \vee, ^*)$ with a strong negation * where $\odot = \wedge$. (In particular, the unit interval $([0,1], \leq, \wedge, \vee, ^*)$ with a strong negation $a^* = 1-a$ for each $a \in [0,1]$) (ref.[12]).
(2) Every continuous t-norm $([0,1], \leq, t, ^*)$ coincided with $\odot = t$ and a strong negation * (ref.[7,21]).
(3) A MV-algebra $(L, \leq, \odot, ^*)$ with a strong negation *.(ref. [7,21])

In this paper, we assume that $(L, \leq, \odot, ^*)$ is a scq-lattice with a strong negation *.

Lemma 2.4. [7,11,21]. For each $x, y, z \in L$, $\{y_i \mid i \in \Gamma\} \subset L$, we have the following properties.
(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$ and $x \odot y \leq x \wedge y$.
(2) $\bigwedge_{i\in\Gamma} y_i^* = (\bigvee_{i\in\Gamma} y_i)^*$ and $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.
(3) If L is a complete MV-algebra, $x \odot (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \odot y_i)$.

All algebraic operations on L can be extended pointwise to the set L^X, where X is a set, as follows: for all $x \in X$ and $\lambda, \mu \in L^X$,
(1) $\lambda \leq \mu$ iff $\lambda(x) \leq \mu(x)$;
(2) $(\lambda \odot \mu)(x) = \lambda(x) \odot \mu(x)$.

Definition 2.5 [8,11]. A function $\mathcal{T} : L^X \to L$ is called an *L-fuzzy topology* on X if it satisfies the following conditions:
(O1) $\mathcal{T}(\overline{1}) = \mathcal{T}(\overline{0}) = 1$.
(O2) $\mathcal{T}(\lambda_1 \odot \lambda_2) \geq \mathcal{T}(\lambda_1) \odot \mathcal{T}(\lambda_2), \forall \lambda_1, \lambda_2 \in L^X$.
(O3) $\mathcal{T}(\bigvee_{i\in\Gamma} \lambda_i) \geq \bigwedge_{i\in\Gamma} \mathcal{T}(\lambda_i), \forall \{\lambda_i\}_{i\in\Gamma} \subset L^X$.
The pair $(X, \mathcal{T})$ is called an *L-fuzzy topological space.*

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be L-fuzzy topological spaces. A function $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is said to be *L-fuzzy continuous* if $\mathcal{T}_2(\mu) \leq \mathcal{T}_1(f^{-1}(\mu))$, $\forall \mu \in L^Y$.

Definition 2.6 [8,11]. A map $\mathcal{I} : L^X \times L_0 \to L^X$ is called an *L-fuzzy interior operator* on X iff $\mathcal{I}$ satisfies the following conditions:
(I1) $\mathcal{I}(\overline{1}, r) = \overline{1}$, $\forall r \in L_0$.
(I2) $\mathcal{I}(\lambda, r) \leq \lambda$, $\forall r \in L_0$.
(I3) If $\lambda \leq \mu$ and $r \leq s$, then $\mathcal{I}(\lambda, s) \leq \mathcal{I}(\mu, r)$.
(I4) $\mathcal{I}(\lambda \odot \mu, r \odot s) \geq \mathcal{I}(\lambda, r) \odot \mathcal{I}(\mu, s)$.
The pair $(X, \mathcal{I})$ is called an *L-fuzzy interior space.*
An L-fuzzy interior space $(X, \mathcal{I})$ is called *topological* if

$$\mathcal{I}(\mathcal{I}(\lambda, r), r) \geq \mathcal{I}(\lambda, r),\ \forall \lambda \in L^X, r \in L_0.$$

Theorem 2.7 [8,11]. *Let $(X, \mathcal{I})$ be an L-fuzzy interior space. Define a map $\mathcal{T}_{\mathcal{I}} : L^X \to L$ by*

$$\mathcal{T}_{\mathcal{I}}(\lambda) = \bigvee\{r \in L \mid \lambda \leq \mathcal{I}(\lambda, r)\}.$$

Then $\mathcal{T}_{\mathcal{I}}$ is an L-fuzzy topology on X induced by $\mathcal{I}$.

3. L-fuzzy topogenous orders and L-fuzzy interior operators

Definition 3.1. A function $\eta : L^X \times L^X \to L$ is called an *L-fuzzy topogenous order* on X, if it satisfies the following axioms: for any $\lambda, \lambda_1, \lambda_2, \mu, \mu_1, \mu_2 \in L^X$,

(T1) $\eta(\underline{1}, \underline{1}) = \eta(\underline{0}, \underline{0}) = 1$,

(T2) if $\eta(\lambda, \mu) \neq 0$, then $\lambda \leq \mu$,

(T3) if $\lambda \leq \lambda_1$, $\mu_1 \leq \mu$ then $\eta(\lambda_1, \mu_1) \leq \eta(\lambda, \mu)$,

(T4) $\eta(\lambda_1 \odot \lambda_2, \mu_1 \odot \mu_2) \geq \eta(\lambda_1, \mu_1) \odot \eta(\lambda_2, \mu_2)$.

Definition 3.2. An L-fuzzy topogenous order η is called *perfect* if

(T5) $\eta(\bigvee_{i\in\Gamma} \lambda_i, \mu) = \bigwedge_{i\in\Gamma} \eta(\lambda_i, \mu)$, for any $\{\mu, \lambda_i \mid i \in \Gamma\} \subset L^X$.

A perfect L-fuzzy topogenous order η is called *biperfect* if

(T6) $\eta(\lambda, \bigwedge_{i\in\Gamma} \mu_i) = \bigwedge_{i\in\Gamma} \eta(\lambda, \mu_i)$, for any $\{\lambda, \mu_i \mid i \in \Gamma\} \subset L^X$.

Theorem 3.3. *Let $\eta_1, \eta_2 : L^X \times L^X \to L$ be L-fuzzy topogenous orders on X. Define the composition $\eta_1 \circ \eta_2$ of η_1 and η_2 on X by*

$$\eta_1 \circ \eta_2(\lambda, \mu) = \bigvee_{\nu\in L^X} (\eta_1(\lambda, \nu) \odot \eta_2(\nu, \mu)).$$

Then $\eta_1 \circ \eta_2$ is an L-fuzzy topogenous order on X.

Proof. Let $\eta_1, \eta_2 : L^X \times L^X \to L$ be L-fuzzy topogenous orders on X.

(T1) and (T3) are easy.

(T2) If $\eta_1 \circ \eta_2(\lambda, \mu) \neq 0$, then there exists $\nu \in L^X$ such that

$$\eta_1 \circ \eta_2(\lambda, \mu) \geq \eta_1(\lambda, \nu) \odot \eta_2(\nu, \mu) \neq 0.$$

So, $\eta_1(\lambda, \nu) \neq 0$ and $\eta_2(\nu, \mu) \neq 0$. It implies $\lambda \leq \nu \leq \mu$.

(T4) It is proved from:

$$\begin{aligned}
&(\eta_1 \circ \eta_2)(\lambda_1, \mu_1) \odot (\eta_1 \circ \eta_2)(\lambda_2, \mu_2) \\
&= \Big(\bigvee_{\rho_1\in L^X} (\eta_1(\lambda_1, \rho_1) \odot \eta_2(\rho_1, \mu_1)) \Big) \odot \Big(\bigvee_{\rho_2\in L^X} (\eta_1(\lambda_2, \rho_2) \odot \eta_2(\rho_2, \mu_2)) \Big) \\
&= \bigvee_{\rho_1,\rho_2\in L^X} \Big((\eta_1(\lambda_1, \rho_1) \odot \eta_1(\lambda_2, \rho_2)) \odot (\eta_2(\rho_1, \mu_1) \odot \eta_2(\rho_2, \mu_2)) \Big) \\
&\leq \bigvee_{\rho_1,\rho_2\in L^X} (\eta_1(\lambda_1 \odot \lambda_2, \rho_1 \odot \rho_2) \odot \eta_2(\rho_1 \odot \rho_2, \mu_1 \odot \mu_2)) \\
&\leq \bigvee_{\nu\in L^X} (\eta_1(\lambda_1 \odot \lambda_2, \nu) \odot \eta_2(\nu, \mu_1 \odot \mu_2)) \\
&\leq \eta_1 \circ \eta_2(\lambda_1 \odot \lambda_2, \mu_1 \odot \mu_2).
\end{aligned}$$

In the next, We introduce the relationship among L-fuzzy topogenous and L-fuzzy interior operators.

Theorem 3.4. *Let η be an L-fuzzy topogenous order on X. Define a function $\mathcal{I}_\eta : L^X \times L_0 \to L^X$ as:*

$$\mathcal{I}_\eta(\lambda, r) = \bigvee \{\mu \in L^X \mid \eta(\mu, \lambda) \geq r\}.$$

Then $\mathcal{I}_\eta$ is an L-fuzzy interior operator on X.

Proof. (1) (I1) Since $\eta(\overline{1}, \overline{1}) = 1$, $\mathcal{I}_\eta(\overline{1}, r) = \overline{1}$.

(I2) Since $\eta(\mu, \lambda) \neq 0$, $\mu \leq \lambda$ implies $\mathcal{I}_\eta(\lambda, r) \leq \lambda$.

(I3) If $\lambda \leq \mu$ and $r \leq s$, since $\eta(\gamma, \mu) \geq \eta(\gamma, \lambda) \geq s \geq r$, then $\mathcal{I}_\eta(\lambda, s) \leq \mathcal{I}_\eta(\mu, r)$.

(I4) From (T4), we have:

$$
\begin{aligned}
&\mathcal{I}_\eta(\lambda, r) \odot \mathcal{I}_\eta(\mu, s) \\
&= \Big\{ \bigvee \{\gamma_1 \in L^X \mid \eta(\gamma_1, \lambda) \geq r\} \Big\} \odot \Big\{ \bigvee \{\gamma_2 \in L^X \mid \eta(\gamma_2, \mu) \geq s\} \Big\} \\
&= \bigvee \{\gamma_1 \odot \gamma_2 \in L^X \mid \eta(\gamma_1, \lambda) \geq r, \eta(\gamma_2, \mu) \geq s\} \\
&\leq \bigvee \{\gamma_1 \odot \gamma_2 \in L^X \mid \eta(\gamma_1 \odot \gamma_2, \lambda \odot \mu) \geq r \odot s\} \\
&\leq \mathcal{I}_\eta(\lambda \odot \mu, r \odot s).
\end{aligned}
$$

Theorem 3.5. *Let η be an L-fuzzy topogenous operator on X. Define a map $\mathcal{T}_{\mathcal{I}_\eta} : L^X \to L$ by*

$$\mathcal{T}_{\mathcal{I}_\eta}(\lambda) = \bigvee \{r \in L \mid \mathcal{I}_\eta(\lambda, r) \geq \lambda\}.$$

Then $\mathcal{T}_{\mathcal{I}_\eta}$ is an L-fuzzy topology on X induced by η.

Proof. It is similarly proved as Theorem 2.7.

Example 3.6. Let X be a set. Define two functions $\eta_0, \eta_1 : L^X \times L^X \to L$ as follows:

$$\eta_0(\lambda, \rho) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \rho = \overline{1}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\eta_1(\lambda, \rho) = \begin{cases} 1, & \text{if } \lambda \leq \rho, \\ 0, & \text{otherwise.} \end{cases}$$

(1) Since $\lambda_1 \odot \lambda_2 \neq \overline{0}$ and $\rho_1 \odot \rho_2 \neq \overline{1}$ imply $\lambda_1 \neq \overline{0}$ and $\lambda_2 \neq \overline{0}$ and $\rho_1 \neq \overline{1}$ or $\rho_2 \neq \overline{1}$, we have

$$\eta_0(\lambda_1 \odot \lambda_2, \rho_1 \odot \rho_2) \geq \eta_0(\lambda_1, \rho_1) \odot \eta_0(\lambda_2, \rho_2).$$

Other cases are easy. Hence η_0 is a biperfect L-fuzzy topogenous order on X.

(2) Since $\lambda_1 \leq \rho_1$ and $\lambda_2 \leq \rho_2$ implies $\lambda_1 \odot \lambda_2 \leq \rho_1 \odot \rho_2$, we have

$$\eta_1(\lambda_1 \odot \lambda_2, \rho_1 \odot \rho_2) \geq \eta_1(\lambda_1, \rho_1) \odot \eta_1(\lambda_2, \rho_2).$$

Other cases are easy. Hence η_1 is a biperfect L-fuzzy topogenous order on X.

(3) We can obtain $\mathcal{I}_{\eta_0}, \mathcal{I}_{\eta_1} : L^X \times L_0 \to L$ as follows:

$$\mathcal{I}_{\eta_0}(\lambda, r) = \begin{cases} \overline{1}, & \text{if } \lambda \in \{\overline{0}, \overline{1}\}\ r \in L_0, \\ \overline{0}, & \text{otherwise,} \end{cases}$$

$$\mathcal{I}_{\eta_1}(\lambda, r) = \lambda, \ \ \forall \lambda \in L^X, \ r \in L_0.$$

(4) We can obtain L-fuzzy topologies $\mathcal{T}_{\mathcal{I}_{\eta_0}}, \mathcal{T}_{\mathcal{I}_{\eta_1}} : L^X \to L$ as follows:

$$\mathcal{T}_{\mathcal{I}_{\eta_0}}(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ 0, & \text{otherwise.} \end{cases} \quad \mathcal{T}_{\mathcal{I}_{\eta_1}}(\lambda) = 1, \forall \lambda \in L^X$$

Example 3.7. Let X be a set. Define a function $\eta : L^X \times L^X \to L$ as follows:

$$\eta(\lambda,\rho) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \rho = \overline{1}, \\ \inf\lambda \wedge \inf\rho, & \text{if } \overline{0} \neq \lambda \leq \rho \neq \overline{1}, \\ 0, & \text{otherwise,} \end{cases}$$

(1) Then η is an L-fuzzy topogenous order on X from:

$$\begin{aligned} \eta(\lambda_1 \odot \lambda_2, \rho_1 \odot \rho_2) &= \inf(\lambda_1 \odot \lambda_2) \wedge \inf(\rho_1 \odot \rho_2) \\ &\geq (\inf(\lambda_1) \odot \inf(\lambda_2)) \wedge (\inf(\rho_1) \odot \inf(\rho_2)) \\ &\geq (\inf\lambda_1 \wedge \inf\rho_1) \odot (\inf\lambda_2 \wedge \inf\rho_2) \\ &= \eta(\lambda_1,\rho_1) \odot \eta(\lambda_2,\rho_2). \end{aligned}$$

Other cases are easy.

(2) We can obtain $\mathcal{I}_\eta : L^X \times L_0 \to L$ as follows:

$$\mathcal{I}_\eta(\lambda, r) = \begin{cases} \overline{0}, & \text{if } \lambda = \overline{0}, r \in L_0 \\ \overline{1}, & \text{if } \lambda = \overline{1}, r \in L_0 \\ \lambda, & \text{if } 0 < r \leq \inf\lambda. \end{cases}$$

(3) We can obtain an L-fuzzy topology $\mathcal{T}_{\mathcal{I}_\eta} : L^X \to L$ as follows:

$$\mathcal{T}_{\mathcal{I}_\eta}(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \inf\lambda & \text{otherwise.} \end{cases}$$

4. L-fuzzy topogenous order and L-fuzzy topologies

In the next, We introduce the relationship among L-fuzzy topogenous and L-fuzzy interior operators.

Theorem 4.1. *Let η be a perfect L-fuzzy topogenous order on X. Define a function $\mathcal{T}_\eta : L^X \to L$ by $\mathcal{T}_\eta(\lambda) = \eta(\lambda,\lambda)$. Then we have the following properties:*

(1) $\mathcal{T}_\eta$ is an L-fuzzy topology on X.

(2) If L is an order dense chain, then $\mathcal{T}_\eta = \mathcal{T}_{\mathcal{I}_\eta}$.

Proof. (1) (O1) From (T1), clearly $\mathcal{T}_\eta(\overline{0}) = \mathcal{T}_\eta(\overline{1}) = 1$.

(O2) For any $\lambda_1, \lambda_2 \in L^X$, we have

$$\begin{aligned} \mathcal{T}_\eta(\lambda_1 \odot \lambda_2) &= \eta(\lambda_1 \odot \lambda_2, \lambda_1 \odot \lambda_2) \\ &\geq \eta(\lambda_1,\lambda_1) \odot \eta(\lambda_2,\lambda_2) \\ &= \mathcal{T}_\eta(\lambda_1) \odot \mathcal{T}_\eta(\lambda_2). \end{aligned}$$

(O3) For each family $\{\lambda_j \mid j \in J\} \subset L^X$, we obtain

$$\begin{aligned} \mathcal{T}_\eta(\bigvee_j \lambda_j) &= \eta(\bigvee_j \lambda_j, \bigvee_j \lambda_j) \\ &= \bigwedge_j \eta(\lambda_j, \bigvee_j \lambda_j) \\ &\geq \bigwedge_j \eta(\lambda_j,\lambda_j) \\ &= \bigwedge_j \mathcal{T}_\eta(\lambda_j). \end{aligned}$$

Thus $\mathcal{T}_\eta$ is an L-fuzzy topology.

(2) Since $\mathcal{T}_\eta(\lambda) = \eta(\lambda, \lambda)$, by Theorem 3.4, $\mathcal{I}_\eta(\lambda, \eta(\lambda, \lambda)) \geq \lambda$. From Theorem 3.5, $\mathcal{T}_{\mathcal{I}_\eta}(\lambda) \geq \eta(\lambda, \lambda) = \mathcal{T}_\eta(\lambda)$. Hence $\mathcal{T}_{\mathcal{I}_\eta} \geq \mathcal{T}_\eta$.

Suppose $\mathcal{T}_{\mathcal{I}_\eta} \not\leq \mathcal{T}_\eta$. Since L is an order dense chain, there exist $\rho \in L^X$ and $r \in L_0$ such that

$$\mathcal{T}_{\mathcal{I}_\eta}(\rho) > r > \mathcal{T}_\eta(\rho) = \eta(\rho, \rho).$$

From the definition of $\mathcal{T}_{\mathcal{I}_\eta}$, there exists $r_1 \in L_1$ with $\mathcal{I}_\eta(\rho, r_1) \geq \rho$ such that

$$\mathcal{T}_{\mathcal{I}_\eta}(\rho) \geq r_1 > r > \eta(\rho, \rho).$$

Since $\rho = \mathcal{I}_\eta(\rho, r_1) = \bigvee\{\mu \mid \eta(\mu, \lambda) \geq r_1\}$, we have

$$\eta(\rho, \rho) = \eta(\mathcal{I}_\eta(\rho, r_1), \rho) = \bigwedge \eta(\mu, \lambda) \geq r_1.$$

It is a contradiction. Hence $\mathcal{T}_{\mathcal{I}_\eta} \leq \mathcal{T}_\eta$.

Definition 4.2. Let (X, η_1) and (Y, η_2) be L-fuzzy topogenous spaces. A function $f : (X, \eta_1) \to (Y, \eta_2)$ is said to be *L-fuzzy topogenous continuous* if

$$\eta_2(\lambda, \mu) \leq \eta_1(f^{-1}(\lambda), f^{-1}(\mu)),\ \forall \lambda, \mu \in L^Y.$$

Theorem 4.3. *Let $(X, \eta_1), (Y, \eta_2)$ and (Z, η_3) be L-fuzzy topogenous spaces. If $f : (X, \eta_1) \to (Y, \eta_2)$ and $g : (Y, \eta_2) \to (Z, \eta_3)$ are L-fuzzy topogenous continuous, then $g \circ f : (X, \eta_1) \to (Z, \eta_3)$ is L-fuzzy topogenous continuous.*

Proof. It follows that, for each $\lambda, \mu \in I^Z$,

$$\begin{aligned} \eta_1((g \circ f)^{-1}(\lambda), (g \circ f)^{-1}(\mu)) &= \eta_1(f^{-1}(g^{-1}(\lambda)), f^{-1}(g^{-1}(\lambda))) \\ &\geq \eta_2(g^{-1}(\lambda), g^{-1}(\mu)) \\ &\geq \eta_3(\lambda, \mu). \end{aligned}$$

Theorem 4.4. *Let (X, η_1) and (Y, η_2) be L-fuzzy topogenous spaces. Let $f : (X, \eta_1) \to (Y, \eta_2)$ be topogenous continuous. Then it satisfies the following statements:*

(1) $f^{-1}(\mathcal{I}_{\eta_2}(\mu, r)) \leq \mathcal{I}_{\eta_1}(f^{-1}(\mu), r)$, for each $\mu \in L^Y$.

(2) $f : (X, \mathcal{T}_{\mathcal{I}_{\eta_1}}) \to (Y, \mathcal{T}_{\mathcal{I}_{\eta_2}})$ is fuzzy continuous.

Proof. (1)

$$\begin{aligned} f^{-1}(\mathcal{I}_{\eta_2}(\mu, r)) &= f^{-1}(\bigvee\{\rho \in L^Y \mid \eta_2(\rho, \mu) \geq r\}) \\ &= \bigvee\{f^{-1}(\rho) \in L^X \mid \eta_2(\rho, \mu) \geq r\} \\ &\leq \bigvee\{f^{-1}(\rho) \in L^X \mid \eta_1(f^{-1}(\rho), f^{-1}(\mu)) \geq r\} \\ &\leq \bigvee\{\lambda \in L^X \mid \eta_1(\lambda, f^{-1}(\mu)) \geq r\} \\ &= \mathcal{I}_{\eta_1}(f^{-1}(\mu), r). \end{aligned}$$

(2) From (1), $\mathcal{I}_{\eta_2}(\mu, r) = \mu$ implies $\mathcal{I}_{\eta_1}(f^{-1}(\mu), r) = f^{-1}(\mu)$. It is easily proved from Theorem 2.7.

Theorem 4.5. *Let $(X, \mathcal{T})$ be an L-fuzzy topology on X. Define a function $\eta_{\mathcal{T}} : L^X \times L^X \to L$ as follows:*

$$\eta_{\mathcal{T}}(\lambda, \rho) = \begin{cases} \bigvee\{\mathcal{T}(\gamma) \mid \gamma \in \Phi_{\lambda,\rho}\} & \text{if } \Phi_{\lambda,\rho} \neq \emptyset, \\ 0 & \text{if } \Phi_{\lambda,\rho} = \emptyset \end{cases}$$

where $\Phi_{\lambda,\rho} = \{\gamma \in L^X \mid \lambda \leq \gamma \leq \rho\}$.

Then we have the following properties:

(1) $\eta_{\mathcal{T}}$ is an L-fuzzy topogenous order on X.

(2) If L is a completely distributive lattice, then $\eta_{\mathcal{T}}$ is perfect.

(3) If η is a perfect L-fuzzy topogenous on X, then $\eta \geq \eta_{\mathcal{T}_\eta}$.

(4) If η is a perfect L-fuzzy topogenous on X and L is order dense, then $\eta \geq \eta_{\mathcal{T}_{\mathcal{I}_\eta}}$.

(5) If L is an order dense chain, then $\mathcal{T}_{\mathcal{I}_{\eta_{\mathcal{T}}}} = \mathcal{T}$.

(6) If L is a completely distributive lattice, then $\mathcal{T}_{\eta_{\mathcal{T}}} = \mathcal{T}$.

Proof. (1) (T1) and (T3) are obvious.

(T2) If $\lambda \not\leq \rho$, then $\Phi_{\lambda,\rho} = \emptyset$ implies $\eta_{\mathcal{T}}(\lambda, \rho) = 0$.

(T4) If $\Phi_{\lambda_1,\rho_1} = \emptyset$ or $\Phi_{\lambda_2,\rho_2} = \emptyset$, then

$$\eta_{\mathcal{T}}(\lambda_1 \odot \lambda_2, \rho_1 \odot \rho_2) \geq \eta_{\mathcal{T}}(\lambda_1, \rho_1) \odot \eta_{\mathcal{T}}(\lambda_2, \rho_2).$$

Let $\Phi_{\lambda_1,\rho_1} \neq \emptyset$ and $\Phi_{\lambda_2,\rho_2} \neq \emptyset$. There exist $\nu_i \in L^X$ with $\lambda_i \leq \nu_i \leq \rho_i$, $i = 1, 2$. It implies $\lambda_1 \odot \lambda_2 \leq (\nu_1 \odot \nu_2) \leq \rho_1 \odot \rho_2$ such that

$$\mathcal{T}(\nu_1) \odot \mathcal{T}(\nu_2) \leq \mathcal{T}(\nu_1 \odot \nu_2).$$

Thus, we have

$$\begin{aligned} &\eta_{\mathcal{T}}(\lambda_1, \rho_1) \odot \eta_{\mathcal{T}}(\lambda_2, \rho_2) \\ &= \Big\{ \bigvee\{\mathcal{T}(\nu_1) \mid \nu_1 \in \Phi_{\lambda_1,\rho_1}\}\Big\} \odot \Big\{ \bigvee\{\mathcal{T}(\nu_2) \mid \nu_2 \in \Phi_{\lambda_2,\rho_2}\}\Big\} \\ &= \bigvee\{\mathcal{T}(\nu_1) \odot \mathcal{T}(\nu_2) \mid \nu_1 \in \Phi_{\lambda_1,\rho_1}, \nu_2 \in \Phi_{\lambda_2,\rho_2}\} \\ &\leq \bigvee\{\mathcal{T}(\nu_1 \odot \nu_2) \mid \nu_1 \in \Phi_{\lambda_1,\rho_1},\ \nu_2 \in \Phi_{\lambda_2,\rho_2}\} \\ &\leq \bigvee\{\mathcal{T}(\nu) \mid \nu \in \Phi_{\lambda_1 \odot \lambda_2, \rho_1 \odot \rho_2}\} \\ &= \eta_{\mathcal{T}}(\lambda_1 \odot \lambda_2, \rho_1 \odot \rho_2). \end{aligned}$$

(2) (T5) For each $\nu_j \in L^X$ with $\lambda_j \leq \nu_j \leq \rho$, we have $\bigvee_j \lambda_j \leq \bigvee_j \nu_j \leq \rho$ such that

$$\eta_{\mathcal{T}}(\bigvee_j \lambda_j, \rho) \geq \mathcal{T}(\bigvee_j \nu_j) \geq \bigwedge_j \mathcal{T}(\nu_j).$$

Hence

$$\bigwedge_j \eta_{\mathcal{T}}(\lambda_j, \rho) = \bigwedge_j \Big(\bigvee\{\mathcal{T}(\nu_j) \mid \nu_j \in \Phi_{\lambda_j,\rho}\}\Big)$$

(since L is a completely distributive lattice)

$$\begin{aligned} &= \bigvee \Big(\bigwedge_j \{\mathcal{T}(\nu_j) \mid \nu_j \in \Phi_{\lambda_j,\rho}\}\Big) \\ &\leq \bigvee\{\mathcal{T}(\bigvee_j \nu_j) \mid \bigvee_j \nu_j \in \Phi_{\bigvee_j \lambda_j,\rho}\}. \\ &\leq \eta_{\mathcal{T}}(\bigvee_j \lambda_j, \rho) \end{aligned}$$

(3) Since $\eta(\lambda,\rho) \geq \eta(\gamma,\gamma)$ for $\lambda \leq \gamma \leq \rho$, we have:

$$\begin{aligned}\eta_{\mathcal{T}_\eta}(\lambda,\rho) &= \bigvee\{\mathcal{T}_\eta(\gamma) \mid \lambda \leq \gamma \leq \rho\}\\ &= \bigvee\{\eta(\gamma,\gamma) \mid \lambda \leq \gamma \leq \rho\}\\ &\leq \eta(\lambda,\rho).\end{aligned}$$

(4) It follows from (3) and Theorem 4.1(2).

(5) Suppose $\mathcal{T}_{\mathcal{I}_{\eta_\mathcal{T}}} \not\geq \mathcal{T}$. Since L is an order dense chain, there exist $\lambda \in L^X$ and $r \in L$ such that

$$\mathcal{T}_{\mathcal{I}_{\eta_\mathcal{T}}}(\lambda) < r \leq \mathcal{T}(\lambda).$$

Since $\mathcal{T}(\lambda) \geq r$, we have $\eta_\mathcal{T}(\lambda,\lambda) \geq \mathcal{T}(\lambda) \geq r$. So, $\mathcal{I}_{\eta_\mathcal{T}}(\lambda,r) \geq \lambda$. Thus,$\mathcal{T}_{\mathcal{I}_{\eta_\mathcal{T}}}(\lambda) \geq r$. It is a contradiction. Thus, $\mathcal{T}_{\mathcal{I}_{\eta_\mathcal{T}}} \geq \mathcal{T}$.

Suppose $\mathcal{T}_{\mathcal{I}_{\eta_\mathcal{T}}} \not\leq \mathcal{T}$. Since L is an order dense chain, there exists $\lambda \in L^X$ with $\mathcal{I}_{\eta_\mathcal{T}}(\lambda,s) = \lambda$ such that

$$\mathcal{T}_{\mathcal{I}_{\eta_\mathcal{T}}}(\lambda) \geq s > r > \mathcal{T}(\lambda).$$

Since $\lambda = \mathcal{I}_{\eta_\mathcal{T}}(\lambda,s) = \bigvee\{\rho_i \mid \eta_\mathcal{T}(\rho_i,\lambda) \geq s\}$, by the definition of $\eta_\mathcal{T}$, for each i, there exists γ_i with $\rho_i \leq \gamma_i \leq \lambda$ such that $\mathcal{T}(\gamma_i) \geq s_i > r$. Thus, $\lambda = \bigvee_i \rho_i \leq \bigvee_i \gamma_i \leq \lambda$ implying that $\lambda = \bigvee_i \gamma_i$. So,

$$\mathcal{T}(\lambda) = \mathcal{T}(\bigvee_i \gamma_i) \geq \bigwedge_i \mathcal{T}(\gamma_i) \geq \bigwedge_i s_i \geq r.$$

It is a contradiction. Thus, $\mathcal{T}_{\mathcal{I}_{\delta_\mathcal{T}}} \leq \mathcal{T}$.

(6) For each L-fuzzy topology $\mathcal{T}$ on X, since L is a completely distributive lattice, by (2), $\eta_\mathcal{T}$ is perfect. By Theorem 4.1, $\mathcal{T}_{\eta_\mathcal{T}}$ is an L-fuzzy topology on X. Since $\eta_\mathcal{T}(\lambda,\lambda) = \bigvee\{\mathcal{T}(\rho) \mid \lambda \leq \rho \leq \lambda\} = \mathcal{T}(\lambda)$, we have

$$\mathcal{T}_{\eta_\mathcal{T}}(\lambda) = \eta_\mathcal{T}(\lambda,\lambda) = \mathcal{T}(\lambda).$$

Theorem 4.7. *Let $(X,\mathcal{I})$ be an L-fuzzy interior space. Define a function $\eta_\mathcal{I} : L^X \times L^X \to L$ as follows:*

$$\eta_\mathcal{I}(\lambda,\rho) = \begin{cases} \bigvee\{r \in L \mid \lambda \leq \mathcal{I}(\rho,r)\}, & \text{if } \lambda \leq \mathcal{I}(\rho,r)\\ 0, & \text{if } \lambda \not\leq \mathcal{I}(\rho,r).\end{cases}$$

Then we have the following properties:

(1) $\eta_\mathcal{I}$ is an L- fuzzy topogenous order on X.

(2)If L is an order dense chain , then $\eta_\mathcal{I}$ is perfect.

(3) $\mathcal{I}_{\eta_\mathcal{I}}(\lambda,r) \leq \mathcal{I}(\lambda,r)$ and $\mathcal{I}_{\eta_\mathcal{I}}(\lambda,s) \geq \mathcal{I}(\lambda,r)$ for each $\lambda \in L^X$, $r,s \in L$ with $s < r$. If L is a chain, $\mathcal{I}_{\eta_\mathcal{I}}(\lambda,r) \leq \mathcal{I}(\lambda,r)$, for each $\lambda \in L^X$, $r \in L$.

(4) If $\mathcal{I}$ is topological and L is an order dense chain , then $\eta_{\mathcal{T}_\mathcal{I}} = \eta_\mathcal{I}$.

Proof. (1) (T1) and (T3) are obvious.

(T2) If $\eta_\mathcal{I}(\lambda,\rho) \neq 0$, there exists $r \in L$ such that $\lambda \leq \mathcal{I}(\rho,r) \leq \rho$.

(T4) Since $\lambda_1 \leq \mathcal{I}(\mu_1,r)$ and $\lambda_2 \leq \mathcal{I}(\mu_2,s)$ imply

$$\lambda_1 \odot \lambda_2 \leq \mathcal{I}(\mu_1,r) \odot \mathcal{I}(\mu_2,s) \leq \mathcal{I}(\mu_1 \odot \mu_2, r \odot s),$$

we have,

$$\begin{aligned}&\eta_\mathcal{I}(\lambda_1,\mu_1) \odot \eta_\mathcal{I}(\lambda_2,\mu_2)\\ &= \bigvee\{r \in L \mid \lambda_1 \leq \mathcal{I}(\mu_1,r)\} \odot \bigvee\{s \in L \mid \lambda_2 \leq \mathcal{I}(\mu_2,s)\}\\ &\leq \bigvee\{r \odot s \in L \mid \lambda_1 \odot \lambda_2 \leq \mathcal{I}(\mu_1 \odot \mu_2, r \odot s)\}\\ &= \bigvee\{r_0 \in L \mid \lambda_1 \odot \lambda_2 \leq \mathcal{I}(\mu_1 \odot \mu_2, r_0)\}\\ &= \eta_\mathcal{I}(\lambda_1 \odot \lambda_2, \mu_1 \odot \mu_2)\end{aligned}$$

(2) Suppose there exists a family $\{\lambda_i \mid i\in\Gamma\}$ such that

$$\eta_{\mathcal{I}}(\bigvee_{i\in\Gamma}\lambda_i,\mu)\not\geq\bigwedge_{i\in\Gamma}\eta_{\mathcal{I}}(\lambda_i,\mu).$$

Since L is an order dense chain, there exists $r\in L$ such that

$$\eta_{\mathcal{I}}(\bigvee_{i\in\Gamma}\lambda_i,\mu)<r<\bigwedge_{i\in\Gamma}\eta_{\mathcal{I}}(\lambda_i,\mu).$$

Since $\eta_{\mathcal{I}}(\lambda_i,\mu)>r$, for each $i\in\Gamma$, there exists $s_i\in L$ such that $s_i>r$ with $\lambda_i\leq\mathcal{I}(\mu,s_i)$. Put $s=\bigwedge_{i\in\Gamma}s_i$. Then $\bigvee_{i\in\Gamma}\lambda_i\leq\mathcal{I}(\mu,s)$,i.e. $\eta_{\mathcal{I}}(\bigvee_{i\in\Gamma}\lambda_i,\mu)\geq s\geq r$. It is a contradiction.

(3) Since $\mathcal{I}(\rho,r)\leq\mathcal{I}(\rho,r)$, then $\eta_{\mathcal{I}}(\mathcal{I}(\rho,r),\rho)\geq r$. Hence, $\mathcal{I}_{\eta_{\mathcal{I}}}(\lambda,r)\geq\mathcal{I}(\lambda,r)$.

Let L be a chain. Since $\mathcal{I}_{\eta_{\mathcal{I}}}(\lambda,r)=\bigvee\{\rho_i\mid\eta_{\mathcal{I}}(\rho_i,\lambda)\geq r\}$, for $s<r$, there exists $s_i\in L$ such that $s<s_i\leq r$ with $\rho_i\leq\mathcal{I}(\lambda,s_i)$. Put $s=\wedge_{i\in\Gamma}s_i\geq s$. Then $\vee_{i\in\Gamma}\rho_i\leq\mathcal{I}(\lambda,s)$. Hence $\mathcal{I}_{\eta_{\mathcal{I}}}(\lambda,r)\leq\mathcal{I}(\lambda,s)$.

(4) Suppose $\eta_{\mathcal{T}_{\mathcal{I}}}\not\geq\eta_{\mathcal{I}}$. Since L is an order dense chain, there exist $r\in L$, $\lambda,\rho\in L^X$ such that

$$\eta_{\mathcal{T}_{\mathcal{I}}}(\lambda,\rho)<r<\eta_{\mathcal{I}}(\lambda,\rho).$$

Since $\eta_{\mathcal{I}}(\lambda,\rho)<r$, there exists $s\in L$ with $s\geq r$ such that $\lambda\leq\mathcal{I}(\rho,s)$. Since

$$\lambda\leq\mathcal{I}(\mathcal{I}(\rho,s),s)=\mathcal{I}(\rho,s)\leq\rho,$$

we have $\mathcal{T}_{\mathcal{I}}(\mathcal{I}(\rho,s))\geq s$. It implies

$$\eta_{\mathcal{T}_{\mathcal{I}}}(\lambda,\rho)\leq\mathcal{T}_{\mathcal{I}}(\mathcal{I}(\rho,s))\geq s\geq r.$$

It is a contradiction. Hence $\eta_{\mathcal{T}_{\mathcal{I}}}\geq\eta_{\mathcal{I}}$.

Suppose $\eta_{\mathcal{T}_{\mathcal{I}}}\not\leq\eta_{\mathcal{I}}$. Since L is an order dense chain, there exist $r\in L$, $\lambda,\rho\in L^X$ such that

$$\eta_{\mathcal{T}_{\mathcal{I}}}(\lambda,\rho)>r>\eta_{\mathcal{I}}(\lambda,\rho).$$

Since $\eta_{\mathcal{T}_{\mathcal{I}}}(\lambda,\rho)>r$, there exists $\rho\in L^X$ with $\lambda\leq\rho\leq\rho$ such that

$$\eta_{\mathcal{T}_{\mathcal{I}}}(\lambda,\rho)\geq\mathcal{T}_{\mathcal{I}}(\rho)\geq r..$$

Thus $\mathcal{T}_{\mathcal{I}}(\rho)\geq r$. It implies

$$\lambda\leq\rho\leq\mathcal{I}(\rho,r).$$

Thus $\eta_{\mathcal{I}}(\lambda,\rho)\geq r$. It is a contradiction.

Theorem 4.8. *Let (X,η) be an L-fuzzy topogenous space. Then we have the following properties:*

(1) *If L is a chain, $\eta_{\mathcal{I}_\eta}\geq\eta$.*

(2) *If (X,η) is perfect and L is an order dense chain, then $\eta=\eta_{\mathcal{I}_\eta}\geq\eta_{\mathcal{T}_\eta}$.*

Proof. (1) Let $\eta(\lambda,\rho)\geq r$. Then $\lambda\leq\mathcal{I}_\eta(\rho,r)$. It implies $\eta_{\mathcal{I}_\eta}(\lambda,\rho)\geq r$. Since L is a chain, $\eta_{\mathcal{I}_\eta}\geq\eta$.

(2) Suppose $\eta_{\mathcal{I}_\eta}\not\leq\eta$. Since L is an order dense chain, there exist $\lambda,\rho\in L^X$ and $s\in L$ such that

$$\eta_{\mathcal{I}_\eta}(\lambda,\rho)>s>\eta(\lambda,\rho).$$

Since $\eta_{\mathcal{I}_\eta}(\lambda,\rho)>s$, there exists $r\in L$ with $r>s$ such that $\lambda\leq\mathcal{I}_\eta(\rho,r)$. Since $\mathcal{I}_\eta(\rho,r)=\bigvee\{\mu_i\mid\eta(\mu_i,\rho)\geq r\}$ and (X,η) is perfect, by (T5), we have

$$\eta(\lambda,\rho)\geq\eta(\mathcal{I}_\eta(\rho,r),\rho)\geq\bigwedge\eta(\mu_i,\rho)\geq r>s.$$

It is a contradiction. Thus $\eta\geq\eta_{\mathcal{I}_\eta}$. So, $\eta=\eta_{\mathcal{I}_\eta}$ and $\eta\geq\eta_{\mathcal{T}_\eta}$ from Theorem 4.5(3).

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