.

ON L-FUZZY PROXIMITY SPACES

M.El-Dardery

Department of Mathematics, Faculty of Science, Fayoum University, Fayoum, Egypt

J. Zhang

College of Science, North China University of Technology, Beijing 100041, PR China

Abstract.

In this paper we study L-fuzzy proximity spaces, where L represents a completely distributive lattice.We shall investigate the level decomposition of L-fuzzy proximity on X and the corresponding L-fuzzy proximity continuous maps. In addition, we shall establish the representation theorems of L-fuzzy proximity on X.

Keywords.

L-fuzzy proximity; L-proximity; L-fuzzy proximity continuous map; L-proximity continuous map

1. Introduction

The concept of fuzzy topology was first defined in 1968 by Chang [5] and later redefined in a somewhat different way by Lowen [21] and by Hutton [12]. According to Šostak[28], these definitions, a fuzzy topology is a crisp subfamily of a fmily fuzzy sets and fuzziness in the concept of openness of a fuzzy set has not been considered, which appears to be a drawback in the process of fuzzification of the concept of topological spaces. Therefore, Šostak introduced a new definition of fuzzy topology in 1985 [30], Later on he developed the theory of fuzzy topological spaces in [29]. After that, several authors [11, 22, 26, 29] have reintroduced the same definition and studied fuzzy topological spaces being unaware of Šostak's work.Katsaras [13] introduced fuzzy proximity in $[0,1]$-fuzzy set theory. Subsequently Wang-jin Liu [17], Artico and Moresco [1] extended it into L-fuzzy set theory. F. Bayoumi [4] shows that all initial and final lifts in the category L-PRI of L-proximity spaces of the internal type and hence all initial and final L-proximities of the internal type do exist. In the framework of [34] we have introduced the two papers [7,8] and in the present paper, we study the level decomposition of an L-fuzzy proximity and the corresponding L-fuzzy proximity continuous maps. In addition, we also establish some representation theorems for L-fuzzy proximity on X. The main results of this paper are several representation theorems for L-fuzzy proximity on X, where L represents a completely distributive lattice. Based on the results of this paper, we have also developed representation theorems for the category L-FP which consist of L-fuzzy proximity spaces and L-fuzzy proximity continuous maps.

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2. Preliminaries

Throughout this paper, L represents a completely distributive lattice with the smallest element $\perp$ and the greatest element $\top$, where $\perp \neq \top$. We define $M(L)$ to be the set of all non-zero $\vee$-irreducible (or coprime) elements in L such that $a \in M(L)$ iff $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. Let $P(L)$ be the set of all non-unit prime elements in L such that $a \in P(L)$ iff $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. Finally, let X be a non-empty usual set, and L^X be the set of all L-fuzzy sets on X. For each $a \in L$, let $\underline{a}$ denote a constant-valued L-fuzzy set with a as its value. Let $\underline{\perp}$ and $\underline{\top}$ be the smallest element and greatest element in L^X, respectively. for the empty set $\phi \subset L$, we define $\wedge\phi = \top$ and $\vee\phi = \perp$.

Defnition 2.1[32].

Suppose that $a \in L$ and $A \subset L$.

(1) A is called a maximal family of a if

(a) inf $A = a$,

(b) $\forall B \subset L$, inf $B \leq a$ implies that $\forall x \in A$ there exists $y \in B$ such that $y \leq x$.

(2) A is called a minimal family of a if

(a) sup $A = a$,

(b) $\forall B \subset L$, sup $B \geq a$ implies that $\forall x \in A$ there exists $y \in B$ such that $y \geq x$.

Remark 2.2.[12].

Hutton proved that if L is a completely distributive lattice and $a \in L$, then there exists $B \subset L$ such that

(i) $a = \bigvee B$, and

(ii) if $A \subset L$ and $a = \bigvee A$, then for each $b \in B$ there is a $c \in A$ such that $b \leq c$.

However, if $\forall a \in L$, and if there exists $B \subset L$ satisfying (i) and (ii), then in general L is not a completely distributive lattice. To this end,Wang [30] introduced the following modification of condition (ii),

(ii') if $A \subset L$ and $a \leq \bigvee A$, then for each $b \in B$ there is a $c \in A$ such that $b \leq c$.

Wang proved that a complete lattice L is completely distributive if and only if for each element $a \in L$, there exists $B \subset L$ satisfying (i) and (ii). Such a set B is called a minimal set of a by Wang [31].The concept of maximal family is the dual concept of minimal family, and a complete lattice L is completely distributive if and only if for each element $a \in L$, there exists a maximal family $B \subset L$.

Let $\alpha(a)$ denote the union of all maximal families of a. Likewise, let $\beta(a)$ denote the union of all minimal sets of a. Finally, let $\alpha^*(a) = \alpha(a) \cap P(L)$ and $\beta^*(a) = \beta(a) \cap M(L)$. One can easily see that both $\alpha(a)$ and $\alpha^*(a)$ are maximal sets of a. Likewise, both $\beta(a)$ and $\beta^*(a)$ are minimal sets of a. Also, we have $\alpha(\top) = \phi$. and $\beta(\perp) = \phi$.

Definition 2.3 [12].

An L-fuzzy topology on X is a map $\mathcal{T} : L^X \to L$ satisfying the following three axioms:

(O1) $\mathcal{T}(\underline{\top}) = \top$;

(O2) $\mathcal{T}(\lambda \wedge \rho) \geq \mathcal{T}(\lambda) \wedge \mathcal{T}(\rho), \forall \lambda, \rho \in L^X$.

(O3) $\mathcal{T}(\bigvee_{i\in\Delta}\lambda_i) \geq \bigwedge_{i\in\Delta}\mathcal{T}(\lambda_i), \forall\{\lambda_i\}_{i\in\Delta} \subset L^X$.

The pair $(X,\mathcal{T})$ is called an L-fuzzy topological space. For every $\lambda \in L^X, \mathcal{T}(\lambda)$ is called the degree of openness of the fuzzy subset λ. Just as an L-topology on X is an ordinary subset of L^X, an L-fuzzy topology on X is a fuzzy subset of L^X.

Definition 2.4.

Let $(X,\mathcal{T}_1)$ and $(Y,\mathcal{T}_2)$ be L-fuzzy topological spaces. A map $f:(X,\mathcal{T}_1)\to(Y,\mathcal{T}_2)$ is called L-fuzzy continuous iff

$$\mathcal{T}_2(\rho) \leq \mathcal{T}_1(f^{\leftarrow}(\rho)),\ \forall\rho\in L^Y.$$

where $f^{\leftarrow}(\rho) = \rho\circ f$

Definition 2.5. *(Artico and Moreseco [1], Katsaras [15], Liu [19]).*

An L-proximity on L^X is a subfamily of $L^X\times L^X$ which satisfies, for any $\lambda,\rho,\mu,\gamma\in L^X$, the following conditions:

(P1) $(\underline{\perp},\underline{\top}) \notin \delta$.

(P2) If $\lambda\leq\rho$, $(\lambda,\mu)\in\delta$ *then* $(\rho,\mu)\in\delta$.

(P3) If $(\lambda,\rho)\in\delta$ *then* $(\rho,\lambda)\in\delta$.

(P4) If $(\lambda,\rho\vee\mu)\in\delta$ *then* $(\lambda,\rho)\in\delta$ *or* $(\lambda,\mu)\in\delta$

(P5) If $(\lambda,\rho)\notin\delta$, *there exists* $\gamma\in L^X$ *such that* $(\lambda,\gamma)\notin\delta$ *and* $(\gamma^{'},\rho)\notin\delta$.

(P6) If $(\lambda,\rho)\notin\delta$ *then* $\lambda\leq\rho^{'}$

As in (Shi [26-28] and Wang [32]) we give the following lemma:

Lemma 2.6.

For $a\in L$ and a map $\delta: L^X\times L^X\to L$, we define

$$\delta_{[a]} = \{(A,B)\in L^X\times L^X \mid \delta(A,B)\geq a\}$$

and

$$\delta^{[a]} = \{(A,B)\in L^X\times L^X \mid a\notin\alpha(\delta(A,B))\}$$

Let δ be a map from $L^X\times L^X$ to L and $a,b\in L$. Then

(1) $a\in\beta(b)\Rightarrow\delta_{[b]}\subset\delta_{[a]}$; $a\in\alpha(b)\Rightarrow\delta^{[a]}\subset\delta^{[b]}$.

(2) $a\leq b\Leftrightarrow\beta(a)\subset\beta(b)\Leftrightarrow\beta^*(a)\subset\beta^*(b)\Leftrightarrow\alpha(b)\subset\alpha(a)\Leftrightarrow\alpha^*(b)\subset\alpha^*(a)$.

(3) $\alpha(\bigwedge_{i\in\triangle}a_i) = \bigcup_{i\in\triangle}\alpha(a_i)$ *and* $\beta(\bigvee_{i\in\triangle}a_i) = \bigcup_{i\in\triangle}\beta(a_i)$ *for any sub-family* $\{a_i\}_{i\in\triangle}\subset L$

3. Level decomposition of an L-fuzzy proximity

Definition 3.1[18].

A map $\delta: L^X\times L^X\to L$ is called an L-fuzzy proximity on X if it satisfies the following conditions:

(FP1) $\delta(\underline{\top},\underline{\perp}) = \perp$,

(FP2) If $\lambda\leq\rho$, *then* $\delta(\lambda,\mu)\leq\delta(\rho,\mu)$.

(FP3) $\delta(\lambda,\rho) = \delta(\rho,\lambda)$.

(FP4) $\delta(\lambda,\rho\vee\mu)\le\delta(\lambda,\rho)\vee\delta(\lambda,\mu)$.

(FP5) $\delta(\lambda,\rho)\ge\bigwedge_{\gamma\in L^X}\{\delta(\lambda,\gamma)\vee\delta(\gamma^{'},\rho)\}$.

(FP6) If $\delta(\lambda,\rho)\neq\top$, *then* $\lambda\le\rho^{'}$.

The pair (X,δ) *is said to be an L-fuzzy proximity space. Just as an L-proximity on* X *is an ordinary subset of* $L^X\times L^X$, *an L-fuzzy proximity on* X *is a fuzzy subset of* $L^X\times L^X$.

An L-fuzzy proximity space is called principal provided that

(P) $\delta(\bigvee_{i\in\Delta}\lambda_i,\mu)\le\bigvee_{i\in\Delta}\delta(\lambda_i,\mu)$.

Remark 3.2.

(1) If $\delta:2^X\times 2^X\to I$ *where* $I=[0,1]$ *such that the above conditions hold respectively, we call it a fuzzifying proximity(resp. principal fuzzifying proximity) on* X *in a sense [33].*

(2) We easily show that every L-fuzzy proximity space is a Samanta's fuzzy proximity space [25] and Ghanism's fuzzy proximity space [10].

Theorem 3.3.

Let δ *be a map* $\delta:L^X\times L^X\to L$. *Then the following conditions are equivalent:*

(1) δ *is an L-fuzzy proximity on* X.

(2) $\forall a\in M(L)$, $\delta_{[a]}$ *is an L-proximity on* X.

(3) $\forall a\in L$, $\delta^{[a]}$ *is an L-proximity on* X.

(4) $\forall a\in P(L)$, $\delta^{[a]}$ *is an L-proximity on* X.

proof. (1) ⇒ (2): *this part is obvious.*

(2) ⇒ (1): *(FP1) For each* $a\in M(L)$, *we have* $(\underline{\top},\underline{\perp})\notin\delta_{[a]}$, *and* $\delta(\underline{\top},\underline{\perp})<a$. *Accordingly,*

$$\delta(\underline{\top},\underline{\perp})<\bigwedge\{a\mid a\in M(L)\}=\perp.$$

Thus, $\delta(\underline{\top},\underline{\perp})=\perp$.

(FP2) Let $\lambda,\rho,\mu\in L^X$ *with* $\lambda\le\rho$. *Clearly, when* $\delta(\lambda,\mu)=\perp$, *we have* $\delta(\lambda,\mu)\le\delta(\rho,\mu)$. *Otherwise if* $\delta(\lambda,\mu)>\perp$, *then for each* $\delta(\lambda,\mu)\ge a$, *we have* $(\lambda,\mu)\in\delta_{[a]}$. *Consequently, by (P2), we have* $(\rho,\mu)\in\delta_{[a]}$, *that is,* $\delta(\rho,\mu)\ge a$. *This further implies that*

$$\delta(\rho,\mu)\ge\bigvee\{a\in M(L)\mid\delta(\lambda,\mu)\ge a\}=\delta(\lambda,\mu).$$

(FP3) Let $\lambda,\rho\in L^X$. *For each* $\delta(\lambda,\rho)\ge a$, *we have* $(\lambda,\rho)\in\delta_{[a]}$. *Consequently* $(\rho,\lambda)\in\delta_{[a]}$ *or* $\delta(\rho,\lambda)\ge a$. *This further implies that*

$$\delta(\rho,\lambda)\ge\bigvee\{a\in M(L)\mid\delta(\lambda,\rho)\ge a\}=\delta(\lambda,\rho).$$

The opposite inequality follows, by interchanging λ *and* ρ.

(FP4)Let $\lambda,\rho,\mu\in L^X$. *Clearly, when* $\delta(\lambda,\rho\vee\mu)=\perp$, *we have* $\delta(\lambda,\rho\vee\mu)\le\delta(\lambda,\rho)\vee\delta(\lambda,\mu)$. *Otherwise if* $\delta(\lambda,\rho\vee\mu)>\perp$,*then for each* $\delta(\lambda,\rho\vee\mu)\ge a$, *we have* $(\lambda,\rho\vee\mu)\in\delta_{[a]}$.

Consequently, we have $(\lambda,\rho)\in\delta_{[a]}$ *or* $(\lambda,\mu)\in\delta_{[a]}$ *and so* $\delta(\lambda,\rho)\geq a$ *or* $\delta(\lambda,\mu)\geq a$ *hence* $\delta(\lambda,\rho)\vee\delta(\lambda,\mu)\geq a$. *This further implies that*

$$\delta(\lambda,\rho)\vee\delta(\lambda,\mu)\geq\bigvee\{a\in M(L)\mid a\leq\delta(\lambda,\rho\vee\mu)\}=\delta(\lambda,\rho\vee\mu).$$

(FP5) Let $\lambda,\rho,\gamma\in L^X$.*Clearly, when* $\bigwedge_{\gamma\in L^X}\left\{\delta(\lambda,\gamma)\vee\delta(\gamma^{'},\rho)\right\}=\bot$, *we have* $\delta(\lambda,\rho)\geq\bigwedge_{\gamma\in L^X}\left\{\delta(\lambda,\gamma)\vee\delta(\gamma^{'},\rho)\right\}$. *Otherwise if* $\bigwedge_{\gamma\in L^X}\left\{\delta(\lambda,\gamma)\vee\delta(\gamma^{'},\rho)\right\}>\bot$, *Then for each*

$$\bigwedge_{\gamma\in L^X}\left\{\delta(\lambda,\gamma)\vee\delta(\gamma^{'},\rho)\right\}\geq a.$$

Then for all $\gamma\in L^X$ *where* $\delta(\lambda,\gamma)\vee\delta(\gamma^{'},\rho)\geq a$. *Consequently, we have for all* $\gamma\in L^X$ *where* $\delta(\lambda,\gamma)\geq a$ *or* $\delta(\gamma^{'},\rho)\geq a$ *implies for all* $\gamma\in L^X$, $(\lambda,\gamma)\in\delta_{[a]}$ *or* $(\gamma^{'},\rho)\in\delta_{[a]}$ *and so* $(\lambda,\rho)\in\delta_{[a]}$ *or* $\delta(\lambda,\rho)\geq a$. *This further implies that*

$$\delta(\lambda,\rho)\geq\bigvee\left\{a\in M(L)\mid\bigwedge_{\gamma\in L^X}\left\{\delta(\lambda,\gamma)\vee\delta(\gamma^{'},\rho)\right\}\geq a\right\}=\bigwedge_{\gamma\in L^X}\left\{\delta(\lambda,\gamma)\vee\delta(\gamma^{'},\rho).\right.$$

Thus, $\delta(\lambda,\rho)\geq\bigwedge_{\gamma\in L^X}\left\{\delta(\lambda,\gamma)\vee\delta(\gamma^{'},\rho).\right.$

(FP6)Clearly from (P6).

(1) $\Rightarrow$ (3): *(P1) Since* $\delta(\underline{\top},\underline{\bot})=\bot$, *clearly for each* $a\in L$, *we have* $a\in\alpha(\delta(\underline{\top},\underline{\bot}))$. *Thus* $(\underline{\top},\underline{\bot})\notin\delta^{[a]}$.

(P2)Consider $\lambda\leq\rho$ *and* $(\lambda,\mu)\in\delta^{[a]}$ *then* $a\notin\alpha(\delta(\lambda,\mu))\supset\alpha(\delta(\rho,\mu))$ *and so* $(\rho,\mu)\in\delta^{[a]}$.

(P3) For all $\lambda,\rho\in L^X$, *let* $(\lambda,\rho)\in\delta^{[a]}$. *Hence* $a\notin\alpha(\delta(\lambda,\rho))=\alpha(\delta(\rho,\lambda))$. *Furthermore, since* $a\notin\alpha(\delta(\rho,\lambda))$, *we have* $(\rho,\lambda)\in\delta^{[a]}$.

(P4)For all $\lambda,\rho,\mu\in L^X$, *let* $(\lambda,\rho\vee\mu)\in\delta^{[a]}$. *We have* $a\notin\alpha(\delta(\lambda,\rho\vee\mu))\supset\alpha(\delta(\lambda,\rho)\vee\delta(\lambda,\mu))$ *hence either* $\delta(\lambda,\rho)\leq\delta(\lambda,\mu)$ *then* $a\notin\alpha(\delta(\lambda,\mu))$ *and so* $(\lambda,\mu)\in\delta^{[a]}$ *or* $\delta(\lambda,\rho)\geq\delta(\lambda,\mu)$ *then* $a\notin\alpha(\delta(\lambda,\rho))$ *and so* $(\lambda,\rho)\in\delta^{[a]}$ *and its clearly if* $\delta(\lambda,\rho)=\delta(\lambda,\mu)$.

(P5) For all $\gamma\in L^X$ *with* $(\lambda,\gamma)\in\delta^{[a]}$ *or* $(\gamma^{'},\rho)\in\delta^{[a]}$, *we have* $a\notin\alpha(\delta(\lambda,\rho))$ *or* $a\notin\alpha(\delta(\gamma^{'},\rho)$. *Then*

$$a\notin\alpha(\delta(\lambda,\rho))\cup\alpha(\delta(\gamma^{'},\rho))=\alpha(\delta(\lambda,\rho))\wedge\alpha(\delta(\gamma^{'},\rho)\supset\alpha(\delta(\lambda,\rho))\vee\alpha(\delta(\gamma^{'},\rho)).$$

Hence

$$a\notin\bigcup_{\gamma\in L^X}\alpha(\delta(\lambda,\rho)\vee\delta(\gamma^{'},\rho))=\alpha(\bigwedge_{\gamma\in L^X}(\delta(\lambda,\rho)\vee\delta(\gamma^{'},\rho))\supset\alpha(\delta(\lambda,\rho)).$$

Then $(\lambda,\rho)\in\delta^{[a]}$.

(P6) Consider $\lambda\not\leq\rho^{'}$, *we have* $\delta(\lambda,\rho)=\top$. *Then* $\alpha(\delta(\lambda,\rho)=\alpha(\top)=\phi$. *Hence* $a\notin\alpha(\delta(\lambda,\rho))$. *Thus* $(\lambda,\rho)\in\delta^{[a]}$.

(3) $\Rightarrow$ (4): *this part is obvious.*

(4) $\Rightarrow$ (1): *(FP1) Since* $(\underline{\perp}, \underline{\top}) \notin \delta^{[\perp]}$. *Thus* $\perp \in \alpha(\delta(\underline{\perp}, \underline{\top}))$. *Then*

$$\delta(\underline{\perp}, \underline{\top}) = \bigwedge \alpha^*(\delta(\underline{\perp}, \underline{\top})) = \perp.$$

(FP2) Let $\lambda, \rho, \mu \in L^X$ *with* $\lambda \leq \rho$. *Clearly, when* $\delta(\lambda, \mu) = \perp$, *we have* $\delta(\lambda, \mu) \leq \delta(\rho, \mu)$. *Otherwise if* $\delta(\lambda, \mu) > \perp$, *then for each* $a \in P(L)$ *and* $a \notin \alpha(\delta(\lambda, \mu))$, *we have* $(\lambda, \mu) \in \delta^{[a]}$. *Consequently, we have* $(\rho, \mu) \in \delta^{[a]}$ *or* $a \notin \alpha(\delta(\rho, \mu))$. *Accordingly, we have*

$$\alpha^*(\delta(\lambda, \mu)) \supset \alpha^*(\delta(\rho, \mu)) \text{ or } \delta(\lambda, \mu) \leq \delta(\rho, \mu).$$

(FP3) Let $\lambda, \rho \in L^X$. *For each* $a \in P(L)$ *and* $a \notin \alpha(\delta(\lambda, \rho))$, *we have* $(\lambda, \rho) \in \delta^{[a]}$. *Consequently* $(\rho, \lambda) \in \delta^{[a]}$, *we have* $a \notin \alpha(\delta(\rho, \lambda))$. *Accordingly, we have*

$$\alpha^*(\delta(\lambda, \rho)) \supset \alpha^*(\delta(\rho, \lambda)) \text{ or } \delta(\lambda, \rho) \leq \delta(\rho, \lambda).$$

The opposite inequality follows, by interchanging λ *and* ρ.

(FP4) Let $\lambda, \rho, \mu \in L^X$. *Clearly, when* $\delta(\lambda, \rho \vee \mu) = \perp$, *we have* $\delta(\lambda, \rho) \vee \delta(\lambda, \mu) \geq \delta(\lambda, \rho \vee \mu)$. *Otherwise if* $\delta(\lambda, \rho \vee \mu) > \perp$, *then for each* $a \in P(L)$ *and* $a \notin \alpha(\delta(\lambda, \rho \vee \mu))$, *we have* $(\lambda, \rho \vee \mu) \in \delta^{[a]}$, *and so* $(\lambda, \rho) \in \delta^{[a]}$ *or* $(\lambda, \mu) \in \delta^{[a]}$. *Consequently, we have*

$$a \notin \alpha(\delta(\lambda, \rho) \cup \alpha(\delta(\lambda, \mu) = \alpha((\delta(\lambda, \rho) \wedge (\delta(\lambda, \mu)) \supset \alpha((\delta(\lambda, \rho) \vee (\delta(\lambda, \mu))$$

. *Accordingly, we have*

$$\alpha^*(\delta(\lambda, \rho \vee \mu) \supset \alpha^*(\delta(\lambda, \rho) \vee (\delta(\lambda, \mu)) \text{ or } \delta(\lambda, \rho \vee \mu) \leq \delta(\lambda, \rho) \vee \delta(\lambda, \mu)$$

(FP5) Let $\lambda, \rho, \gamma \in L^X$. *Clearly, when* $\delta(\lambda, \rho) = \perp$, *we have* $\delta(\lambda, \rho) \geq \bigwedge_{\gamma \in L^X} \{\delta(\lambda, \gamma) \vee \delta(\gamma^{'}, \rho)\}$. *Otherwise if* $\delta(\lambda, \rho) > \perp$, *then for each* $a \in P(L)$ *and* $a \notin \alpha(\delta(\lambda, \rho))$ *Hence* $(\lambda, \rho) \in \delta^{[a]}$. *Consequently, there exsits* $\gamma \in L^X$ *where* $(\lambda, \gamma) \in \delta^{[a]}$ *and* $(\gamma^{'}, \rho) \in \delta^{[a]}$ *implies* $a \notin \bigcup_{\gamma \in L^X}(\alpha(\delta(\lambda, \gamma) \vee \delta(\gamma^{'}, \rho)) = \alpha(\bigwedge_{\gamma \in L^X}(\delta(\lambda, \gamma) \vee \delta(\gamma^{'}, \rho))$ *Accordingly, we have*

$$\alpha^*(\delta(\lambda, \rho)) \supset \alpha^*(\bigwedge_{\gamma \in L^X} (\delta(\lambda, \gamma) \vee \delta(\gamma^{'}, \rho))$$

or

$$\delta(\lambda, \rho) \leq \bigwedge_{\gamma \in L^X} (\delta(\lambda, \gamma) \vee \delta(\gamma^{'}, \rho))$$

.

(FP6)Clearly from (P6).

We can now state the following decomposition theorem of L-fuzzy proximity. The proof is straightforward and therefore omitted.

Theorem 3.4..

Let δ be an L-fuzzy proximity on X. Then

$$\delta = \bigvee_{a\in L}(\underline{a}\wedge\delta_{[a]}) = \bigvee_{a\in M(L)}(\underline{a}\wedge\delta_{[a]}) = \bigwedge_{a\in L}(\underline{a}\vee\delta^{[a]}) = \bigwedge_{a\in P(L)}(\underline{a}\vee\delta^{[a]})$$

Corollary 3.5..

Let δ_1 and δ_2 be L-fuzzy proximities on X, then the following conditions are equivalent:

(1) $\delta_1 = \delta_2$.
(2) $\forall a\in L, {\delta_1}_{[a]} = {\delta_2}_{[a]}$.
(3) $\forall a\in M(L), {\delta_1}_{[a]} = {\delta_2}_{[a]}$.
(4) $\forall a\in L, \delta_1^{[a]} = \delta_2^{[a]}$.
(5) $\forall a\in P(L), \delta_1^{[a]} = \delta_2^{[a]}$.

Theorem 3.6.

Let δ be an L-fuzzy proximity on X, then
(1) $a\in L,\ \delta_{[a]} = \bigcap_{b\in\beta(a)}\delta_{[b]}$.
(2) $\forall a\in M(L),\ \delta_{[a]} = \bigcap_{b\in\beta^*(a)}\delta_{[b]}$.
(3) $a\in L,\ \delta^{[a]} = \bigcap_{a\in\alpha(b)}\delta^{[b]}$.
(4) $\forall a\in P(L),\ \delta^{[a]} = \bigcap_{a\in\alpha^*(a),b\in P(L)}\delta^{[b]}$.

Proof.

(1)By Lemma 2.7, we have that $\forall a\in L, \delta_{[a]}\subset\bigcap_{b\in\beta(a)}\delta_{[b]}$. *To show that* $\delta_{[a]}\supset\bigcap_{b\in\beta(a)}\delta_{[b]}$, *we take* $(\lambda,\rho)\in L^X\times L^X$ *and* $(\lambda,\rho)\in\bigcap_{b\in\beta(a)}\delta_{[b]}$. *Notice that* $\forall b\in\beta(a),\delta(\lambda,\rho)\geq b$. *Hence* $\delta(\lambda,\rho)\geq\bigvee\{b\mid b\in\beta(a)\}=a$, *which implies that* $(\lambda,\rho)\in\delta_{[a]}$.

(2) The proof is similar to (1).

(3)By Lemma 2.7, we have that $\forall a\in L,\delta^{[a]}\subset\bigcap_{a\in\alpha(b)}\delta^{[b]}$. *To show that* $\delta^{[a]}\supset\bigcup_{a\in\alpha(b)}\delta^{[b]}$, *we take* $(\lambda,\rho)\in L^X\times L^X$ *and* $(\lambda,\rho)\in\bigcup_{a\in\alpha(b)}\delta^{[b]}$. *Notice that* $\forall b\in L$ *and* $a\in\alpha(b)$, *it follows that* $b\notin\alpha(\delta(\lambda,\rho))$. *We prove by contradiction as follows. Suppose that* $a\in\alpha(\delta(\lambda,\rho))$. *Notice that* $\delta(\lambda,\rho)=\bigwedge\{b\mid b\in\alpha(\delta(\lambda,\rho))\}$ *and* $\alpha(\delta(\lambda,\rho))=\bigcup\{\alpha(b)\mid b\in\alpha(\delta(\lambda,\rho))\}$. *There must exist* $b\in\alpha(\delta(\lambda,\rho))$ *such that* $a\in\alpha(b)$. *But this is impossible.*

(4) The proof is similar to (3).

Remark 3.7.

(1) $b\in\beta(a)$ *implies* $b\ll a$, *where* $b\ll$ *is way-below relation [6], i.e.* $b\ll a$ *if and only if for every up-directed set* S *in* L, $\bigvee S\geq a$ *implies that there exists* $s\in S$ *such that* $s\geq b$;

(2) If $a\in M(L)$, *then* $b\in\beta^*(a)$ *if and only if* $b\ll a$.

(3) $\forall a\in M(L),\ \delta_{[a]}=\bigcap_{b\in\beta^*}\delta_{[b]}\Leftrightarrow\delta_{[a]}=\bigcap_{b\ll a,b\in M(L)}\delta_{[b]}$.

Proof.

(1) Since $\beta(a)$ is a minimal set of a, from Definition 2.1, we have that for every up-directed set S in L, if $\bigvee S \geq a$, then $\forall b \in \beta(a)$ there exists $s \in S$ such that $s \geq b$. It follows that $b \ll a$.

(2) Let $a \in M(L)$ and $b \ll a$. From Theorems 1.3.6 and 1.3.8 in [15] and Definition 2.1, we know that $\beta^*(a)$ is both an up-directed set and a lower set, and $\bigvee \beta^*(a) = a$. Hence, there exists $b^{'} \in \beta^*(a)$ such that $a \geq b^{'} \geq b$. In other words, $b \in \beta^*(a)$. Conversely, if $b \in \beta^*(a)$, then since $\beta^*(a) \subset \beta(a)$ and $b \in \beta^*(a)$ implies $b \in \beta(a)$. It follows that $b \ll a$.

(3) It is obvious.

Theorem 3.8.

Let $\left\{\delta_{[a]} \mid a \in M(L)\right\}$ be a family of L-proximities on X. Then the following conditions are equivalent:

(1) There exists an L-fuzzy proximity δ on X such that $\delta_{[a]} = \delta_a$ for each $a \in M(L)$.

(2) $\forall a \in M(L)$, $\delta_a = \bigcap_{b \in \beta^*(a)} \delta_b$.

Proof. (1) $\Rightarrow$ (2)*: This holds because of Theorem 3.5.*

(2) $\Rightarrow$ (1)*: Let $\delta = \bigvee_{a \in M(L)} (a \wedge \delta_a)$. Obviously, we have $\delta_a \subset \delta_{[a]}$. For any $(\lambda, \rho) \in \delta_{[a]}$, we have $\delta(\lambda, \rho) \geq a$ and $\bigvee \left\{b \in M(L) \mid (\lambda, \rho) \in \delta_b\right\} \geq a$. Next, since $\beta^*(a)$ is a minimal family of a, for each $b \in \beta^*(a)$, there exists $b^{'} \in M(L)$ such that $b \geq b^{'}$ and $(\lambda, \rho) \in \delta_{b'} \subset \delta_b$. Therefore, $\bigcap_{b \in \beta^*(a)} \delta_b = \delta_a$.*

Similarly, we can state the following theorems.

Theorem 3.9. *Let $\left\{\delta_a \mid a \in P(L)\right\}$ be a family of L-proximities on X. Then the following conditions are equivalent:*

(1) There exists an L-fuzzy proximity δ on X such that $\delta^{[a]} = \delta_a$ for each $a \in P(L)$.

(2) $\forall a \in P(L)$, $\delta_a = \bigcap_{a \in \alpha^*(b)} \delta_b$.

Theorem 3.10. *Let $\left\{\delta_a \mid a \in L\right\}$ be a family of L-proximities on X. Then the following conditions are equivalent:*

(1) There exists an L-fuzzy proximity δ on X such that $\delta_{[a]} = \delta_a$ for each $a \in L$.

(2) $\forall a \in L$, $\delta_a = \bigcap_{b \in \beta(a)} \delta_b$.

Theorem 3.11. *Let $\left\{\delta_a \mid a \in L\right\}$ be a family of L-proximities on X. Then the following conditions are equivalent:*

(1) There exists an L-fuzzy proximity δ on X such that $\delta^{[a]} = \delta_a$ for each $a \in L$.

(2) $\forall a \in L$, $\delta_a = \bigcap_{a \in \alpha(b)} \delta_b$.

4. Representation theorems of L-fuzzy proximities

Let $LP[X]$ denote the family of all L-proximities on X. Let $LFP[X]$ denote the family of all L-fuzzy proximities on X. The order relation on $LFP[X]$ is defined as follow:

$$\forall \delta_1, \delta_2 \in LFP[X], \delta_1 \preceq \delta_2 \Leftrightarrow \forall (\lambda, \rho) \in L^X \times L^X, \delta_1(\lambda, \rho) \leq \delta_2(\lambda, \rho).$$

Theorem 4.1.
$(LFP[X], \preceq)$ is a complete lattice. In fact, it is a complete sub-meet-semilattice of $L^{L^X \times L^X}$, i.e. closed under the $\wedge$ of $L^{L^X \times L^X}$.

Proof. *Let X be a set. Define two maps $\delta : L^X \times L^X \to L$ as follows:*

$$\delta_0(\lambda, \rho) = \begin{cases} \bot, & \text{if } \lambda = \overline{\bot} \text{ or } \rho = \overline{\bot}, \\ \top, & \text{otherwise,} \end{cases}$$

$$\delta_1(\lambda, \rho) = \begin{cases} \bot, & \text{if } \lambda \leq \rho^{'}, \\ \top, & \text{otherwise.} \end{cases}$$

Clearly, we have $\delta_0, \delta_1 \in LFP[X]$, and they are the smallest element and the greatest element in $(LFP[X], \preceq^L)$, respectively. Next, let $\{\delta_i \mid i \in \Delta\} \subset LFP[X]$ and $\delta = \bigwedge_{i\in\Delta}^{\preceq^L} \delta_i$. Obvious $\delta \in LFP[X]$. Accordingly, $(LFP[X], \preceq)$ is a complete lattice.

To facilitate further illustration, let us define the following classes:

$$U^L[X] = \{F : L \to LP[X] \mid \forall a \in L, F(a) = \cap_{a\in\alpha(b)} F(b)\}$$

$$U_L[X] = \{F : L \to LP[X] \mid \forall a \in L, F(a) = \cap_{b\in\beta(a)} F(a)\}$$

$$U_{M(L)}[X] = \{F : M(L) \to LP[X] \mid \forall a \in M(L), F(a) = \cap_{b\in\beta^*(a)} F(b)\}$$

$$U_{P(L)}[X] = \{F : P(L) \to LP[X] \mid \forall a \in P(L), F(a) = \cap_{a\in\alpha^*(b)} F(b)\}$$

In addition, let us define the following order relations within the classes $U^L[X]$, $U_L[X]$,$U_{M(L)}[X]$ and $U_{P(L)}[X]$:

$$F_1, F_2 \in U^L[X], F_1 \preceq^L F_2 \Leftrightarrow \forall a \in L, F_1(a) \subset F_2(a)$$

$$F_1, F_2 \in U_L[X], F_1 \preceq_L F_2 \Leftrightarrow \forall a \in L, F_1(a) \subset F_2(a)$$

$$F_1, F_2 \in U_{M(L)}[X], F_1 \preceq_{M(L)} F_2 \Leftrightarrow \forall a \in M(L), F_1(a) \subset F_2(a)$$

$$F_1, F_2 \in U_{P(L)}[X], F_1 \preceq_{P(L)} F_2 \Leftrightarrow \forall a \in P(L), F_1(a) \subset F_2(a)$$

Theorem 4.2.
$(U^L[X], \preceq^L)$, $(U_L[X], \preceq_L)$, $(U_{M(L)}[X], \preceq_{M(L)})$ and $(U_{P(L)}[X], \preceq_{P(L)})$ are complete lattices. Obviously, $U^L[X], \preceq^L)$ and $U_L[X], \preceq_L)$ are complete sub-meet-semilattices of the lattice $(LP[X])^L$ (i.e., closed under the $\wedge$ of $(LP[X])^L$, when $\{F_i \mid i \in \Delta\} \subset U^L[X]$, $F = \bigwedge_{i\in\Delta}^{\preceq_L} F_i$ be defined as $\forall a \in L, F(a) = \bigcap_i \in \Delta F_i(a)$, $(U_{M(L)}[X], \preceq_{M(L)})$ is a complete sub-meet-semilattices of the lattice $(LP[X])^{M(L)}$, and $(U_{P(L)}[X]), \preceq_{p(L)})$ is a complete sub-meet-semilattices of the lattice $(LP[X])^{P(L)}$.

Proof. *$\forall a \in L$, let us define $F_\perp(a) = \{(\lambda, \rho) \mid \lambda \neq \perp, \rho \neq \perp\}$ and $F_\top(a) = \{(\lambda, \rho) \mid \lambda \not\leq \rho'\}$. Clearly, we have $F_\perp(a), F_\top(a) \in U^L[X]$, and they are the smallest element and the greatest element in $(U^L[X], \preceq^L)$, respectively. Next, let $\{F_i \mid i \in \Delta\} \subset U^L[X]$ and $F = \bigwedge_{i\in\Delta}^{\preceq^L} F_i$. Since*

$$F(a) = \bigcap_{i\in\Delta} F_i(a) = \bigcap_{i\in\Delta} \bigcap_{a\in\alpha(b)} F_i(b) = \bigcap_{a\in\alpha(b)} \bigcap_{i\in\Delta} F_i(b) = \bigcap_{a\in\alpha(b)} F(b),$$

it follows that $F \in U^L[X]$. Accordingly, $(U^L[X], \preceq^L)$ is a complete lattice. The same argument can be used to prove the rest of the theorem.

The following representation theorem of L-fuzzy proximity follows naturally.

Theorem 4.3.
The map $f : LFP[X] \to U^L[X], \delta \mapsto F_\delta$ (for every $a \in L$ and $F_\delta(a) = \delta^{[a]}$ is an isomorphism in the category of complete meet-semilattices and $f^{\leftarrow} : U^L[X] \to LFP[X], F \mapsto \delta_F = \bigwedge_{a\in L}(\underline{a} \vee F(a))$.

Proof.
For each $\delta \in LPT[X]$, it is easy to verify that

$$F_\delta(a) = \delta^{[a]} = \bigcap_{a\in\alpha(b)} \delta^{[b]} = \bigcap_{a\in\alpha(b)} F_\delta(b)$$

Hence, $F_\delta \in U^L[X]$. Next, by Theorems 3.3, 3.4 and Corollary 3.5, it suffices to show that f is an injection. Since $(\lambda, \rho) \notin (\delta_F)^{[c]}$ iff

$$\alpha((\delta_F(\lambda, \rho)) = \bigcup_{a\in L} \alpha((\underline{a} \vee F(a))((\lambda, \rho)) = \bigcup \{\alpha(a) \mid a \in L, (\lambda, \rho) \notin F(a)\}$$

iff there exists $a \in L$ such that $c \in \alpha(a)$ and $(\lambda, \rho) \notin F(a)$ iff $(\lambda, \rho) \notin \bigcap_{c\in\alpha(a)} F(a) = F(c)$, we have $F_{\delta_F}(c) = \delta_F^{[c]} = F(c)$. This shows that $F_{\delta_F} = F$. It follows that f is a surjection as well as a bijection, and

$$f^{\leftarrow} : U^L[X] \to LFP[X], F \mapsto \delta_F = \bigwedge_{a\in L} (\underline{a} \vee F(a))$$

Next, let $\delta_1, \delta_2 \in LFP[X]$ and $\{\delta_i \mid i \in \Delta\} \subset LFP[X]$. Then it is straightforward to show that $f(\delta_1) \preceq^L f(\delta_2)$ when $\delta_1 \preceq \delta_2$. Hence $f(\bigwedge_{i\in\Delta} \delta_i) = \bigwedge^{\preceq^L}_{i\in\Delta} f(\delta_i)$ and the proof is complete.

The following Theorem follows directly from the above proof.

Theorem 4.4.
The map $f : LFP[X] \to U_{P(L)}[X], \delta \mapsto F_\delta$ (for every $a \in P(L)$ and $F_\delta(a) = \delta^{[a]}$ is an isomorphism in the category of complete meet-semilattices and $f^{\leftarrow} : U_{P(L)}[X] \to LFP[X], F \mapsto \delta_F = \bigwedge_{a\in P(L)}(\underline{a} \vee F(a))$.

Theorem 4.5.
The map $f : LFP[X] \to U_L[X], \delta \mapsto F_\delta$ (for every $a \in L$ and $F_\delta(a) = \delta_{[a]}$ is an isomorphism in the category of complete meet-semilattices and $f^{\leftarrow} : U^L[X] \to LFP[X], F \mapsto \delta_F = \bigwedge_{a\in L}(\underline{a} \vee F(a))$.

Proof.
For each $\delta \in LPT[X]$, it is easy to verify that

$$F_\delta(a) = \delta_{[a]} = \bigcap_{b\in\beta(a)} \delta_{[b]} = \bigcap_{b\in\beta(a)} F_\delta(b)$$

Hence, $F_\delta \in U_L[X]$. Next, by Theorems 3.4 and Corollary 3.5, it suffices to show that f is an injection. It is proved easily that $(\lambda, \rho) \in (\delta_F)_{[c]}$ iff

$$\delta_F((\lambda, \rho)) = \bigvee_{a\in L} (\underline{a} \wedge F(a))((\lambda, \rho)) = \bigvee \{a \mid (\lambda, \rho) \in F(a)\} \geq c$$

iff (because of Lemma 2.7)

$$\bigcup_{(\lambda,\rho)\in F(a)} \beta(a) = \beta(\bigvee \{a \mid (\lambda, \rho) \in F(a)\}) \supset \beta(c)$$

On the other hand, we can prove

$$(\lambda, \rho) \in F(c) = \bigcap_{a\in\beta(\alpha)} F(a) \Leftrightarrow \forall a \in \beta(\alpha), (\lambda, \rho) \in F(a) \Leftrightarrow \bigcup_{(\lambda,\rho)\in F(a)} \beta(a) \supset \beta(c)$$

Clearly, $\forall a \in \beta(\alpha), (\lambda, \rho) \in F(a) \Rightarrow \bigcup_{(\lambda,\rho)\in F(a)} \beta(a) \supset \bigcup_{a\in\beta(c)} = \beta(c)$. Conversely, for each $d \in \beta(c) \subset \bigcup_{(\lambda,\rho)\in F(a)} \beta(a)$, then there exists $a \in L$ such that $d \in \beta(a)$ and $(\lambda, \rho) \in F(a) = \bigcap_{b\in\beta(a)} F(b)$. It show that $(\lambda, \rho) \in F(d)$. So, we conclude that $(\lambda, \rho) \in (\delta_F)_{[c]} \Leftrightarrow (\lambda, \rho) \in F(c)$, i.e., $F_{\delta_F}(c) = (\delta_F)_{[c]} = F(c)$. This shows that $F_{\delta_F} = F$. It follows that f is a surjection as well as a bijection, and

$$f^{\leftarrow} : U_L[X] \to LFP[X], F \mapsto \delta_F = \bigvee_{a\in L} (\underline{a} \wedge F(a))$$

Next, let $\delta_1, \delta_2 \in LFP[X]$ and $\{\delta_i \mid i \in \Delta\} \subset LFP[X]$. Then it is straightforward to show that $f(\delta_1) \preceq^L f(\delta_2)$ when $\delta_1 \preceq \delta_2$. Hence $f(\bigwedge_{i\in\Delta} \delta_i) = \bigwedge^{\preceq^L}_{i\in\Delta} f(\delta_i)$ and the proof is complete.

The following Theorem follows directly from the above proof.

Theorem 4.6.

The map $f : LFP[X] \to U_{M(L)}[X], \delta \mapsto F_\delta$ (for every $a \in M(L)$ and $F_\delta(a) = \delta_{[a]}$ is an isomorphism in the category of complete meet-semilattices and $f^{\leftarrow} : U_{M(L)}[X] \to LFP[X], F \mapsto \delta_F = \bigvee_{a\in M(L)}(\underline{a} \wedge F(a))$.

5. L-fuzzy continuous proximity maps

Definition 5.1.

Let (X, δ_1) and (Y, δ_2) be two L-fuzzy proximity spaces. Let $f : X \to Y$ be a map. $f : (X, \delta_1) \to (Y, \delta_2)$ is called L-fuzzy proximity continuous map if for every $(\lambda, \rho) \in L^Y \times L^Y$ we have

$$\delta_1(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)) \geq \delta_2(\lambda, \rho),$$

where $f^{\leftarrow}(\lambda) = \lambda \circ f$.

From Definition 5.1, obviously, $f : (X, \delta_1) \to (Y, \delta_2)$ is an L-fuzzy proximity continuous if and only if $\forall a \in M(L)$, $f : (X, \delta_{1_{[a]}}) \to (Y, \delta_{2_{[a]}})$ is an L-proximity continuous map.

Excepting this, we have the followings equivalent conditions:

Theorem 5.2.

Let (X, δ_1) and (Y, δ_2) be L-fuzzy proximity spaces and $f : X \to Y$ be a map. Then the following conditions are equivalent:

(1) $f : (X, \delta_1) \to (Y, \delta_2)$ is an L-fuzzy proximity continuous map.

(2) $\forall a \in M(L)$, $f : (X, \delta_{1_{[a]}}) \to (Y, \delta_{2_{[a]}})$ is an L-proximity continuous map.

(3) $\forall a \in L$, $f : (X, \delta_1^{[a]}) \to (Y, \delta_2^{[a]})$ is an L-proximity continuous map.

(4) $\forall a \in P(L)$, $f : (X, \delta_1^{[a]}) \to (Y, \delta_2^{[a]})$ is an L-proximity continuous map.

Proof. (1) $\Rightarrow$ (2)*: This part is obvious.*

(2) $\Rightarrow$ (1)*: $\forall(\lambda, \rho) \in L^Y \times L^Y$, $a \in M(L)$ such that $a \leq \delta_2(\lambda, \rho)$, we have $(\lambda, \rho) \in \delta_{2_{[a]}}$ and $(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)) \in \delta_{1_{[a]}}$ by the continuity of $f : (X, \delta_{1_{[a]}}) \to (Y, \delta_{2_{[a]}})$. Accordingly, $\delta_1(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)) \geq a$ for each $\forall a \in M(L) \cap M(\delta_2(\lambda, \rho))$, where $M(\delta_2(\lambda, \rho)) = \{a \in M(L) \mid a \leq \delta_2(\lambda, \rho)\}$. It follows that $\delta_1(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)) \geq \bigvee M(\delta_2(\lambda, \rho)) = \delta_2(\lambda, \rho)$.*

(1) $\Rightarrow$ (3)*: $\forall(\lambda, \rho) \in L^Y \times L^Y$, since $\delta_1(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)) \geq \delta_2(\lambda, \rho)$, it follows from Lemma 2.7 that $a \notin \alpha(\delta_1(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)))$ when $\forall a \in L$, if $a \notin \alpha(\delta_2(\lambda, \rho))$. In other words, if $(\lambda, \rho) \in \delta_2^{[a]}$, then $(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)) \in \delta_1^{[a]}$. Thus $f : (X, \delta_1^{[a]}) \to (Y, \delta_2^{[a]})$ is a fuzzy proximity continuous map.*

(3) $\Rightarrow$ (4)*: This is obvious.* (4) $\Rightarrow$ (1)*: For $\forall a \in P(L)$ and $(\lambda, \rho) \in L^Y \times L^Y$, if $a \notin \alpha(\delta_2(\lambda, \rho)$, then $(\lambda, \rho) \in \delta_2^{[a]}$. Thus $(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)) \in \delta_1^{[a]}$ by the continuity of $f : (X, \delta_1^{[a]}) \to (Y, \delta_2^{[a]})$. In other words, $a \notin \alpha(\delta_1(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)))$ and $\alpha^*(\delta_1(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho))) \subset \alpha^*(\delta_2(\lambda, \rho))$. It follows from Lemma 2.7 that*

$$\delta_1(f^{\leftarrow}(\lambda), f^{\leftarrow}(\rho)) \geq \delta_2(\lambda, \rho)$$

Hence the proof is completed.

Definition 5.3.

Let (X, δ_1) and (Y, δ_2) be two L-fuzzy proximity spaces. Let $f : X \to Y$ be a map. $f : (X, \delta_1) \to (Y, \delta_2)$ is called an L-fuzzy proximity homeomorphism if f is bijective and f and $f^{\leftarrow}$ are L-fuzzy continuous maps.

Theorem 5.4.

Let (X, δ_1) and (Y, δ_2) be L-fuzzy proximity spaces and $f : X \to Y$ be a bijective map. Then the following conditions are equivalent:

(1) $f : (X, \delta_1) \to (Y, \delta_2)$ is an L-fuzzy proximity homeomorphism .

(2) $\forall a \in M(L)$, $f : (X, \delta_{1_{[a]}}) \to (Y, \delta_{2_{[a]}})$ is an L-proximity homeomorphism .

(3) $\forall a \in L$, $f : (X, \delta_1^{[a]}) \to (Y, \delta_2^{[a]})$ is an L-proximity homeomorphism .

(4) $\forall a \in P(L)$, $f : (X, \delta_1^{[a]}) \to (Y, \delta_2^{[a]})$ is an L-proximity homeomorphism .

Proof. *It follows from Definitions 5.1, 5.3 and Theorems 5.2 .*

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