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GENERALIZED REVERSE DERIVATIONS ON SEMIPRIME RINGS

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Abstract: We generalize the notion of reverse derivation by introducing generalized reverse derivations. We define an *l-generalized reverse derivation* (*r-generalized reverse derivation*) as an additive mapping $F : R \to R$, satisfying $F(xy) = F(y)x + yd(x)$ $(F(xy) = d(y)x + yF(x))$ for all $x, y \in R$, where d is a reverse derivation of R. We study the relationship between generalized reverse derivations and generalized derivations on an ideal in a semiprime ring. We prove that if F is an l-generalized reverse (or r-generalized) derivation on a semiprime ring R, then R has a nonzero central ideal.

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Keywords: semiprime ring, ideal, derivation, reverse derivation, l-generalized derivation, r-generalized derivation, l-generalized reverse derivation, r-generalized reverse derivation

1. Introduction

Throughout this paper R denotes an associative ring with center $Z(R)$. If I is a subset of R, then $C_R(I)$ denotes the centralizer of I which is defined by

$$C_R(I) = \{x \in R \mid xa = ax \text{ for all } a \in I\}.$$

Recall that R is *prime* if $aRb = (0)$ implies that $a = 0$ or $b = 0$. The ring R is *semiprime* if $aRa = 0$ implies $a = 0$ (obviously, every prime ring is semiprime). As usual, $[x, y]$ denotes the commutator $xy - yx$. We will make extensive use of the basic commutator identities $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$. An additive mapping d from R into itself is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Given $a \in R$, the additive mapping $d : R \to R$ defined by $d(x) = [x, a]$ for all $x \in R$ is a derivation called the *inner derivation* of R determined by a.

The notion of reverse derivation arose in one early paper of Herstein [1], when he studied Jordan derivations on prime associative rings. The notion of reverse derivation has relations with some generalizations of derivations. A reverse derivation is an additive mapping d from a ring R into itself satisfying $d(xy) = d(y)x + yd(x)$ for all $x, y \in R$. So, each reverse derivation is a Jordan derivation (but the converse is not true in general). In the anticommutative case each reverse derivation is an antiderivation, and each antiderivation is a reverse derivation. The reverse derivations in the case of prime Lie and prime Malcev algebras were studied by Hopkins and Filippov. Those papers provided some examples of nonzero reverse derivations for the simple 3-dimensional Lie algebra sl_2 (see [2]) and characterized the prime Lie algebras admitting a nonzero reverse derivation (see [3, 4]). In particular, Filippov proved that each prime Lie algebra, admitting nonzero reverse derivation is a PI-algebra. Filippov also described all reverse derivations of prime Malcev algebras [5]. The supercase of reverse derivations (antisuperderivations) of simple Lie superalgebras was studied by Kaygorodov in [6] and [7]. He proved that every reverse superderivation of a simple finite-dimensional Lie superalgebra over an algebraically closed field of characteristic zero is the zero mapping. After that, Kaygorodov proved that every r-generalized reverse (or l-generalized) derivation of a simple (non-Lie) Malcev algebra is the zero mapping (see [8]).

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Generalized derivations on Jordan ideals in prime rings

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Abstract: Let R be a 2-torsion free prime ring with center $Z(R)$, J be a nonzero Jordan ideal also a subring of R, and F be a generalized derivation with associated derivation d. In the present paper, we shall show that $J \subseteq Z(R)$ if any one of the following properties holds: (i) $[F(u), u] \in Z(R)$, (ii) $F(u)u = ud(u)$, (iii) $d(u^2) = 2F(u)u$, (iv) $F(u^2) - 2uF(u) = d(u^2) - 2ud(u)$, (v) $F^2(u) + 3d^2(u) = 2Fd(u) + 2dF(u)$, (vi) $F(u^2) = 2uF(u)$ for all $u \in J$.

Key words: Prime rings, Jordan ideals, generalized derivations, derivations

1. Introduction

Let R denote an associative ring with center $Z(R)$. For any $x, y \in R$, we write the commutator $[x, y] = xy - yx$, and the Jordan product $x \circ y = xy + yx$. We recall that a ring R is called prime if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$; it is called a semiprime if $aRa = (0)$ implies that $a = 0$. A prime ring is clearly a semiprime ring. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. A ring R is said to be n-torsion free, where $n \neq 0$ is a positive integer, if whenever $na = 0$, with $a \in R$, then $a = 0$. An additive subgroup J is said to be a Jordan ideal of R if $uor \in J$, for all $u \in J$ and $r \in R$. One may observe that every ideal of R is a Jordan ideal of R but the converse need not be true. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U$ and $r \in R$. It is clear that if $charR = 2$, then the Jordan ideal and Lie ideal of R are the same. In [4] Huang proved: Let R be an associative prime ring with $\operatorname{char} R \neq 2$, U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$, and F a generalized derivation associated with $d \neq 0$. If any one of the following conditions holds: (1) $[d(x), F(y)] = 0$, (2) $d(x) \circ F(y) = 0$, (3) either $d(x) \circ F(y) = x \circ y$ or $d(x) \circ F(y) + x \circ y = 0$, (4) either $[d(x), F(y)] = [x, y]$ or $[d(x), F(y)] + [x, y] = 0$, (5) either $[d(x), F(y)] = (x \circ y)$ or $[d(x), F(y)] + (x \circ y) = 0$, (6) either $d(x) \circ F(y) = [x, y]$ or $d(x) \circ F(y) + [x, y]$, (7) either $d(x) \circ F(y) + xy \in Z(R)$ or $d(x) \circ F(y) - xy \in Z(R)$for all $x, y \in U$, then either $d = 0$ or $U \subseteq Z(R)$.

Motivated by the results of Huang, we continue this line of investigation. In this paper, we study generalized derivation F with derivation d if any one of the following conditions holds: (i) $[F(u), u] \in Z(R)$, (ii) $F(u)u = ud(u)$, (iii) $d(u^2) = 2F(u)u$, (iv) $F(u^2) - 2uF(u) = d(u^2) - 2ud(u)$, (v) $F^2(u) + 3d^2(u) = 2Fd(u) + 2dF(u)$, (vi) $F(u^2) = 2uF(u)$ for all u in a Jordan ideal that is also a subring of R.

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ORIGINAL PAPER

Generalized derivations on Lie ideals in semiprime rings

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Abstract Herstein (J Algebra 14:561–571, 1970) proved that given a semiprime 2-torsion free ring R and an inner derivation d_t, if $d_t^2(U) = 0$ for a Lie ideal U of R then $d_t(U) = 0$. Carini (Rend Circ Mat Palermo 34:122–126, 1985) extended this result for an arbitrary derivation d, proving that $d^2(U) = 0$ implies $d(U) \subseteq Z(R)$. The aim of this paper is to extend the results mentioned above for right (resp. left) generalized derivations. Precisely, we prove that if R admits a right generalized derivation F associated with a derivation d such that $F^2(U) = (0)$, then $d^3(U) = (0)$ and $(d^2(U))^2 = (0)$. Furthermore, if F is also a left generalized derivation on U, then $d(U) = F(U) = (0)$, and $d(R)$, $F(R) \subseteq C_R(U)$. On the other hand, if (F, d), (G, g) are, respectively, right and left generalized derivations that satisfy $F(u)v = uG(v)$ for all $u, v \in U$, then $d(U)$, $g(U) \subseteq C_R(U)$.

Keywords Semiprime ring · Lie ideal · Derivation · Generalized derivation

Mathematics Subject Classification 16W25 · 16N60 · 16U80

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Orthogonality of two left and right generalized derivations on ideals in semiprime rings

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Abstract
In this paper, we present some results concerning orthogonality of l-generalized derivations and r-generalized derivations on ideals in semiprime rings, we also study the connections between orthogonality and some properties of the composition of a l-generalized derivation and a r-generalized derivation. These results are related to results of Brešar and Vukman (Rad Mat 5(2):237–246, 1989), that extend a theorem by Posner (Proc Am Math Soc 8:1093–1100, 1957) about products of derivations on prime rings.

Keywords Semiprime ring · Ideal · Left generalized derivation · Right generalized derivation · Derivation · Orthogonal generalized derivation

Mathematics Subject Classification Primary 16W25 · 16N60 · 16U80

1 Introduction

Throughout this paper R denotes an associative ring. Recall that R is prime if $aRb = (0)$ implies that $a = 0$ or $b = 0$. R is semiprime if $aRa = (0)$ implies $a = 0$ (Obviously, every prime ring is semiprime). If A is a subset of R, $l(A)$ denotes the left annihilator of A, defined by $l(A) = \{x \in R \mid xa = 0 \quad \text{for all } a \in A\}$. An additive map $d : R \to R$ is called derivation if it satisfies $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. For a fixed $a \in R$, the map $I_a : R \to R$ given by $I_a(x) = [a, x] = ax - xa$ is a derivation which is called inner derivation. An additive map $F : R \to R$ is called l-generalized derivation if there exist a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$ (notion introduced by Brešar [3]). It is

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