Faculty of Science
Department of Mathematics

Results on ring and algebra derivations and applications to Banach algebras

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By

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Approval Sheet

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Results on ring and algebra derivations and applications to Banach algebras

The main goal of the current research topics is studying particular classes of rings and algebras that satisfy certain types of identities involving several kinds of derivations. Our dissertation contains five chapters.

Chapter 1 includes main definitions and concepts that will be used in the present dissertation. It displays the concepts of derivations and its various kinds. Also we mention definitions of some classes of rings. Moreover, it includes some well-known facts on prime and semiprime rings which are necessary to illustrate the topics in the following chapters. Also, it contains four sections of the pioneering and motivating results for the following chapters. We wish by this chapter to make our dissertation to be as independent as possible.

Chapter 2 is devoted to study the relationship between the generalized derivations associated with Hochschild 2-cocycles and generalized Jordan triple derivations associated with Hochschild 2-cocycles. We showed the following results:

(1) If a ring R is a 2-torsion free semiprime, then every generalized Jordan triple derivation on R associated with a Hochschild 2-cocycle β is a generalized derivation associated with β if one of the following holds (i) β is symmetric. (ii)There are $x, y, z \in R$ with $T(x, y, z) = xyz - zyx$ is a nonzero divisor.

(2) If R is a prime ring with charactristic not two, S a non-central square closed Lie ideal of R and f is an additive mapping of R such that $f(pkp) = f(p)kp + pf(k)p + pkf(p) + \beta(p, k)p + \beta(pk, p)$ for all $p, k \in S$, where β is a Hochschild 2-cocycle, then $f(pk) = f(p)k + pf(k) + \beta(p, k)$ for all $p, k \in S$.

In **chapter 3**, we give the concept of centrally-extended left derivations, provide some examples that insure the existance of these maps and prove some related results to this new concept as follows.

(i) If R is a semiprime ring admitting a CE-left derivation β, then $\beta(z) \in Z$(center of R) for all $z \in Z$.

(ii) If R is a semiprime ring admitting a CE-left derivation β, then $\beta(m) \in M(R)$ for all $m \in M(R)$, where $M(R) = \{m \in Z : mR \,\square\, Z\}$.

(iii) Every CE-left derivation β of a ring R that has no nonzero central ideals is additive.

(iv) If R is a semiprime ring with no nonzero central ideals admitting a CE-left derivation β which preservses the multiplication of Z by R, then β is a left derivation.

(v) If R is a semiprime ring admitting a CE-left derivation β and $\beta(0) \neq 0$, then R contains a nonzero central ideal.

(vi) If R is a semiprime ring admitting a CE-left derivation β such that $[\beta(h), \beta(p)] = [h, p]$ for all $h, p \in R$ or $[\beta(h), \beta(p)] = -[h, p]$ for all $h, p \in R$, then R is commutative.

(vii) If R is a prime ring with characteristic not two admitting a nonzero CE-left derivation β such that $[\beta(h), h] \in Z$ for all $h \in R$, then R is commutative.

(viii) If R is a prime ring, $char(R) \neq 2$ and β is a nonzero CE-left derivation such that $\beta(h \circ p) \in Z$ for all $h, p \in R$, then R is commutative.

In chapter 4 we give the concept of centrally-extended Jordan endomorphisms (CE-Jordan endomophism), gave some examples that insure the existance of these mappings and to show every CE-Jordan endomophism is not Jordan endomophism, in general. Also, we showed the following results:

(i) Let R be a non-commutative prime ring of characteristic not two and center Z. If R admits a CE-Jordan epimorphism G such that $[G(x), x] \in Z$ for all $x \in R$, then R is an order in a central simple algebra of dimension at most 4 over its center or $G(x) = \mu x$ for all $x \in R$, where $\mu \in C$ (the extended centroid of R).

(ii) Let R be a non-commutative prime ring of characteristic not two with an involution * and center Z. If R admits a CE-Jordan epimorphism G such that $[G(x), x^*] \in Z$ for all $x \in R$, then R is an order in a central simple algebra of dimension at most 4 over its center or $G(x) = \mu x^*$ for all $x \in R$, where $\mu \in C$ (the extended centroid of R).

Chapter 5 studies the commutativity of prime rings and Banach algebras with homoderivations. Our main results are:

(1) Let R be a prime ring with characteristic not 2 and center z. If R admits a homoderivation f, then R is commutative when one of these conditions holds (i) $f([a^m, b^m]) - ([a^m, b^m]) \in z$, (ii) $f([a^m, b^m]) \in z$, (iii) $f(a^m \circ b^n) - (a^m \circ b^n)] \in Z$,

(iv) $f(a^m \circ b^n) \in z$ for all $a, b \in R$, where m and n are fixed positive integers.

Also, we proved the following:

(2) Let A be a prime Banach algebra with center Z(A), T_1 and T_2 be non-empty open subsets of A. If f is a non-zero continuous linear homoderivation of A, then A is commutative when one of these conditions holds

(i) $f([a^m, b^m]) - ([a^m, b^m]) \in Z(A)$ or $f(a^m \circ b^m) - (a^m \circ b^m)] \in Z(A)$ for all $a \in T_1$, $b \in T_2$.

(ii) $f([a^m, b^m]) \in Z(A)$ or $f(a^m \circ b^m) \in Z(A)$ for all $a \in T_1$, $b \in T_2$.