جامعة الفيوم
Fayoum university
Faculty of Science
Mathematics Department

INDUCED AND COINDUCED REPRESENTATIONS OF HOPF GROUP COALGEBRA

By

Mahmoud Mohammed El-Sofy

A Thesis Submitted in partial fulfillment

Of

The requirements for the degree of

Doctor of Philosophy

In

Pure Mathematics

(Algebra)

Department of Mathematics

Faculty of Science, Fayoum

Fayoum University

2007

INDUCED AND COINDUCED REPRESENTATIONS OF HOPF GROUP COALGEBRA

By

Mahmoud Mohammed El-Sofy Ali

Prof. Dr. Kamal Ahmed Dib

Prof. of Mathematics, Supervisor of the faculty of computers and information, Fayoum University.

Signature " "

Prof. Dr. Ahmed Sadek Hegazi

Prof. of Mathematics, Department of mathematics, Faculty of Science, Mansoura University.

Signature " "

Dr. Fatma M. Ismail

Doctor of Mathematics, Department of mathematics, Faculty of Science, Cairo University.

Signature "

INDUCED AND COINDUCED REPRESENTATIONS OF HOPF GROUP COALGEBRA

Abstract

In this thesis, we define a procedure for induced and coinduced representations of Hopf group coalgebra. This procedure is given from two points of view, the first is by group coisotropic quantum subgroup and Hopf group subcoalgebra and the other is given from subHopf group coalgebra. The geometric realization of such induced representations is explained. By constructing example of induction of specified Hopf group coalgebra the simplicity theory of such induction is studied.

INTRODUCTION

The theory of representation is one of the fundamental algebraic structures in algebra theory. It plays an important role in the theoretical physics and mathematical physics beside its algebraic importance; we refer to [4, 11, 13] for details.

The theory of quantum group began its development in about 1982-1985. It is now 20 years since 1986 ICM address V. G. Drinfeld (define the quantum group as quasitriangular Hopf algebra) ignited a wild frenzy of research activity in this area and things related to it. It is connected with many other parts of mathematics, old and new, and remains an area of active, fruitful research today. Many authors began to study the algebraic properties of quantum groups for example [15, 16]. The induced representation of a quantum group is introduced by [5, 8, 10, 12, 17]. Most of these results are:

• In 1992, A Gonzalez-Ruiz and L. A. Ibort [10] gave a framework for induction of quantum group representation:

Let (B, σ) be a quantum subgroup of a quantum group H and ρ be a left corepresentation of B on the space V, then $\Delta \otimes I$ defines a left H-comodule on

$$Ind_R(\rho) = \{F \in H \otimes V : (R \otimes I)F = (I \otimes \rho)F\}$$

where $R = (I \otimes \sigma)\Delta$. Also, $I \otimes \Delta$ defines a right H-comodule on

$$Ind_L(\rho) = \{F \in V \otimes H : (I \otimes L)F = (\rho \otimes I)F\}$$

where $L = (\sigma \otimes I)\Delta$.

These two comodules, called induced representation, are not necessarily equivalent.

- In 1998, Nicola Ciccoli [8] studied the relation between induced representation from coisotropic quantum subgroup and quantum embeddable homogeneous space.

Let H be a Hopf algebra and (C, σ) left coisotropic quantum subgroup of H. If V is left C-comodule by θ, then

$$Ind(\theta) = \{F \in H \otimes V : (R \otimes I)F = (I \otimes \theta)F\}$$

is left H-comodule by $\Delta \otimes I$, this comodule $\mathrm{Ind}(\theta)$ is called induced representation by V. If (C, σ) has a section φ then

1. $\mathrm{Ind}(\theta) \cong (B^C \otimes V)$ as left B^C-module
2. $H \cong (B^C \otimes C)$ as vector space, where

$$B^C = \{h \in H : (I \otimes \sigma)\Delta(h) = h \otimes \sigma(1)\}$$

and by section φ we means $\varphi \in \mathrm{Hom}(C, H)$ convolution invertible such that

a. $\phi(\sigma(1)) = 1$

b. $\sigma(\phi(c)_1 u) \otimes \phi(c)_2 = \sigma(v_1 u) \otimes \phi\sigma(v_2)$

for all $c \in C, u \in H, v \in \sigma^{-1}(c)$.

In 2001, Hegazi, Agawany, Fatma Ismail and Ibrahim [12] proved that:

1. If H is quantum group and (B, σ) a quantum subgroup of H with V is left B-module by φ, then

$$
\begin{aligned}
V' &= \{h \otimes l : R(h) \otimes \phi(b \otimes l) \\
&= h \otimes 1 \otimes \phi(b \otimes l) \text{ for all } b \in B\}
\end{aligned}
$$

is a first type Hopf representation of H.

2. Let H be a Hopf algebra, B subHopf algebra of H and V right B-comodule. Then V ⊗ H ⊗ H is right Hopf module over H.

3. Let H be a Hopf algebra and B subHopf algebra of H . Then we have

a. If B is isolated subHopf algebra of H, then there exists a Hopf epimorphism $\sigma : H \to B$ such that (B, σ) is a Hopf subalgebra of H.

b. If (B, σ) is a Hopf subalgebra of H and $\sigma^2 = \sigma$, then B is an isolated subHopf algebra of H.

Recently, Hopf group coalgebra is introduced by turaev [20]. Virelizier [21] studied the algebraic properties of the Hopf group coalgebras and also in his work [21] has studied quasitriangular Hopf algebra to the setting of Hopf group coalgebra. Zunino [23] gave the dual structure of Hopf group coalgebra.

The main goal of this thesis is to study the induced representation of Hopf group coalgebra. We restrict our attention to finite dimensional case of Hopf group coalgebra. To reach this goal we need to define the concepts of subHopf group coalgebra and Hopf group subcoalgebra, and also group quantum coisotropic subgroup. Also, since the structure f Hopf group coalgebras is new, we defined some definitions for this structure in section 1.2 and we gave some results for dual of Hopf group coalgebra in section 1.3.

This thesis consists of four chapters. Chapter one is divided into three sections, in section one we collected the basic definitions, theorems and some examples of algebra, coalgebra and Hopf algebra to be able to deal with Hopf algebra theory. And we gave the definition of quantum groups and dual quasitriangular bialgebra or Hopf algebra and we also explained the main different between the definitions of quantum subgroup and subquantum group. In section two, we gave the structure of Hopf group coalgebra and some of its basic properties [20, 21]. And we added some definition for this structure and some results. In section three we gave the definition of dual of Hopf group coalgebra [23] and we added some definitions for this concept and some results.

Chapter two is divided into three sections, in the first two sections we introduced basic concepts of induced and coinduced representations of algebra and Hopf algebra which will be useful in what follows. In the other section our induction procedure is explained taking into consideration the new structure of Hopf group coalgebra.

In chapter three, the purpose is to show that the induced representation Ind(ρ) from Hopf group subcoalgebra of H is isomorphic as module to the tensor product of π −quantum embeddable homogeneous space B with the given comodule V. In case (C, σ) is left π −coisotropic quantum subgroup, then H is isomorphic to $C \otimes G$ as a vector space where $G = \{G_\alpha\}_{\alpha\in\pi}$, where

$$G_\alpha = \{h \in H_\alpha : L_{1,\alpha}(h) = (\sigma_1 \otimes I_\alpha^H)\Delta_{1,\alpha}^H(h) = \sigma_1(1) \otimes h\}.$$

In chapter four, we have studied the simplicity of our induced representations of Hopf π −coalgebra. Let H be a Hopf π −coalgebra, (C, σ) a π−coisotropic quantum subgroup. If we have two equivalent right π−comodules over C, what is the relation between the two induced representations (and also what is the relation between the two coinduced representations) over H ? Also, we have the following question: what is the algebraic relation of the induction with respect to direct sums of representations in point of view of simplicity. Finally, we have proved that if the induced representation is simple, then its algebraic induction is constructed from simple representation of the group coisotropic quantum subgroup. If we have simple representation of group coisotropic quantum subgroup for Hopf π−coalgebra, then the induced representation is not necessary to be simple.

In this thesis, unless otherwise, K is a field and K-space means vector space over K. A map f from a space V into a space W always means linear map over K. The tensor product $V \otimes W$ is understood to be $V \otimes_K W$, $I : V \to V$ always denotes the identity map, and the transposition map $\tau : V \otimes W \to W \otimes V$ is defined by $\tau(v \otimes w) = w \otimes v$ for $v \in V, w \in W$. Let $f : C \to D$ be a map. Then $f^* : D^* \to C^*$ is a map, where $f^*(\phi)(c) = \phi(f(c))$ for all $\phi \in D^*, c \in C$.