On Fuzzifying Topological Structures

Thesis

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By

Mostafa El-Dardery Ahmed Hussein

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Promotors

Dr. M-S. Saif

Faculty of Science, Faiyum
Cairo University-Egypt

Dr. S. N. EL-Deeb

Faculty of Science, Faiyum
Cairo University-Egypt

Dr. M. K. EL-Gayyar

Faculty of Engineering, Port-Said, Suez-Canal Universit

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Preface

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Most of the predicates dealt with in our daily life and in modern science are imprecise. Examples are the predicates, "good", "bad", "very tall", etc. The imprecision which concerns us here is not statistical but is an intrinsic concept in such predicates, which we call "Fuzzy" predicates. A fuzzy predicate is one that cannot be defined precisely for each object, because some objects have an ambigrops (fuzzy) status in that regard.

Multiple-valued logic has, for over 50 years, been providing theories which can deal with the logic of fuzziness. Its departure from ordinary logic has been allowing statements to take truth values in the closed unit interval I = [0, 1], rather than in the set {0, 1} only. The understanding, in some of those theories, has been that when the truth values of statements are compared, the higher such a value is, the higher the "credibility" of the corresponding statement becomes.

However, set theory has been late in keepping peace with logic, Prof. Lotfi Zadeh took the required step in 1965 [44] which is now among the classics of science. His idea has been to allow membership values of elements (from a certain universal set of discourse X) in a "fuzzy subset" A in X, to be real numbers from the closed unit interval [0, 1].

Subsequently in 1967 Goguen suggested the more general theory of L-fuzzy sets [15]. In this theory, the membership values in fuzzy sets can be from any lattice L [*], rather than [0, 1] in particular. As Zadeh's intention has been, fuzzy sets open wider scopes for a multitude of applied sciences. Among these are control, optimization, information theory, data bases, decision theory, artifical intelligence, biology, medical diagnosis, sociology and economics.

On the other hand scientists have started to color most domains of classical mathematics such as: topology, algebraic structures, relation theory, number theory, differential calculus, measure theory, etc.

In 1968, C. L. Chang [11] has introduced the concept of fuzzy topological spaces based on a straightforward generalization of union and intersection to fuzzy sets.

In 1979, Katsaras [20] created the concept of a fuzzy proximity spaces which has been developed in other papers [5].

Recently, many outhors such as: Abd El-Monsef [1, 2], Azad [5, 6], Hutton [17-19], Kerre [29-31] lowen [,], Pu-Pao-ming and Liu Ying - Ming [39, 40] and others have further extended.

In 1986, Badard [12] intreoduced the basic idea of smooth structures and he defined a smooth topological spaces.

In 1990, MingShing - Ying [33-35] introduced a concept of fuzzifying topology and in [36] he introduced a concept of fuzzifying uniform spaces.

In 1992, A. A. Ramadan [42], R. Badard, A. A. Ramadan and A. S. Mashhour [8], investigated some properties of smooth topological spaces and smooth peruniform spaces. M. K. EL-Gayyar [14], investigated also smooth topological spaces and some application of the smoothness structures.

In this thesis we use the concept of a fuzzifying topology and the concpet of fuzzifying uniform space to introduce and study the new concepts of a fuzzifying proximity spaces and a fuzzifying syntopogenous structure. This thesis includes a preface, four chapters 1-4, c and a list of refrences.

In chapter 1, we recall most of definitions and results needed in the sequel about fuzzy sets; proximity, uniformity, and syntopogenous;

multiple-valued logic and fuzzifying topology. Some properties are established.

In chapter 2, we study the basic concepts of a fuzzifying uniform space. Some results have been added to complete the idea of this new concept.

In chapter 3, we introduce and study the new concepts of a fuzzifying proximity space. Also relations between a fuzzifying uniform space and a fuzzifying proximity space are investigated.

In chapter 4, we introduce and study the new concepts of a fuzzifying syntopogenous structure. Also relations between these new concepts are investigated.

Note.

(i) The main results of chapter (3) are conditionally accepted for publication in "International Journal of Fuzzy Math."

(ii) The main results of chapter (4) are submitted for publication in "International Journal of Fuzzy Math."

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Chapter I

Introduction

1.1 Fuzzy Sets And Operations On Fuzzy Sets

In 1965, Zadeh [44] introduced the idea of a fuzzy set as an extension of classical set theory. In classical set theory, an element either belongs to a set, or does not belong to a set while Zadeh's definition of a fuzzy set is, A fuzzy set A in some universe X is a mapping from X into the closed unit interval [0, 1]. For $x \in X$, A(x) means the degree of membership of x in A ; A(x) = 1 means full-membership, A(x) = 0 means non membership and $A(x) \in]0, 1[$ denotes partial membership. We denote by I of [0, 1] and may be replaced by arbitrary complete distributive lattice with order reversing involution [15]. So if L' denotes by any lattice then a X-L mapping is said to be an L-fuzzy set. More details on L-fuzzy sets can be found in [19], The collection of all fuzzy sets in X will be denoted by I^X [44]. The case I={0,1}:=2 is essentially set- theory, for elements of 2^X are characteristic functions and called "crisp" fuzzy sets.

Here, we recall most of definitions and results needed in the sequel about operations on fuzzy sets.

Definition 1.1.1 [31]

1) The union $A \cup B$ of A and B as the fuzzy set

$$A \cup B : X \to [0, 1]$$
$$x \to \max(A(x), B(x)) \qquad \forall x \in X$$

2) The intersection $A \cap B$ of A and B as the fuzzy set

$$A \cap B : X \to [0, 1]$$
$$x \to \min(A(x), B(x)) \qquad \forall x \in X$$

3) The complement A^c of A as the fuzzy set

$$A^c : X \to [0, 1]$$

$$x \to 1 - A(x) \qquad \forall x \in X$$

4) The corresponding partial order relation $\subset$ on I^X is defined for arbitrary A and B in I^X as

$$A \subset B \Leftrightarrow (\forall x \in X)\,(A(x) \leq B(x))$$

Definition 1.1.2 [31]

1) The weak α-level (α-cut), where $\alpha \in]0, 1]$, of a fuzzy set A on X is denoted A_α and is defined as :

$$A_\alpha = \{x : x \in X \text{ and } A(x) \geq \alpha\}$$

2) The strong α-level (α-cut), where $\alpha \in [0, 1[$, of a fuzzy set A on X is denoted A_α and is defined as :

$$A_\alpha = \{x : x \in X \text{ and } A(x) > \alpha\}$$

3) The height of a fuzzy set A on X is defined as :

$$\text{height}(A) = \sup_{x \in X} A(x),$$

and A is normal if, height (A) = 1

Proposition 1.1.3 [31]

Let A and B be fuzzy sets on X

1) If $0 < \alpha < \beta$ then $A_\beta \subset B_\alpha$

2) If $A \subset B$ then $A_\alpha \subset B_\alpha$, $\forall \alpha \in [0,1[$

3) $A = B \Leftrightarrow (\forall \alpha \in]0,1])(A_\alpha = B_\alpha)$

$\Leftrightarrow (\forall \alpha \in [0,1[)(A_\alpha = B_\alpha)$

4) $(A \cap B)_\alpha = A_\alpha \cap B_\alpha \quad \forall\, \alpha \in \,]0,1]$

$(A \cap B)_\alpha = A_\alpha \cap B_\alpha \quad \forall\, \alpha \in [0,1]$

5) $(A \cup B)_\alpha = A_\alpha \cup B_\alpha \quad \forall\, \alpha \in \,]0,1]$

$(A \cup B)_\alpha = A_\alpha \cup B_\alpha \quad \forall\, \alpha \in \,]0,1]$

Definition 1.1.4 [30]

Union and intersection can be extended to arbitrary families $(A_i)_{i \in I}$ of fuzzy sets A_i in X, as follows:

1. $\left(\bigcup_{i \in I} A_i\right)(x) = \sup_{i \in I} A_i(x) \qquad \forall x \in X.$

2. $\left(\bigcap_{i \in I} A_i\right)(x) = \inf_{i \in I} A_i(x) \qquad \forall x \in X.$

Propsition 1.1.5 [30, 39, 40]

Let A, $B \in I^X$ and $\{A_i : i \in I\} \subset I^X$; then :

1. $A \cup B = B \cup A$ and $A \cap B = B \cap A$

2. $A \cup A = A$ and $A \cap A = A$

3. $A \cup \left(\bigcap_{i \in I} A_i\right) = \bigcap_{i \in I}(A \cup A_i)$ and $A \cap \left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I}(A \cap A_i)$

4. $\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$ and $\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I}(A_i^c)$

Proposition 1.1.6 [5, 12, 30, 38]

Let $f : X \to Y$, $g : Y \to Z$, $A \in I^X$, $B \in I^Y$, $C \in I^Z$, $\{A_i : i \in I\} \subset I^X$ and $\{B_i : i \in I\} \subset I^Y$. Then,

1. $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$

2. $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$

3. $f^{-1}(B^c) = (f^{-1}(B))^c$

4. $f(f^{-1}(B)) \subset B$, with equality if f is surjective.

5. $A \subset f^{-1}(f(A))$., with equality if f is injective.

6. $f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i)$

7. $f^{-1}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f^{-1}(B_i)$

8. $f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I}(f(A_i))$

9. $f\left(\bigcap_{i\in I} A_i\right) \subset \bigcap_{i\in I} f(A_i)$

10. $(gof)^{-1}(C) = f^{-1}(g^{-1}(C))$

11. $(gof)(A) = g(f(A))$.

Definition 1.1.7 [5]

Let $A \in I^X$ and $B \in I^Y$. Then the cartesian product $A\times B$ is a fuzzy set in $X\times Y$, defined as follows :

$$(A \times B)(x,y) = \min(A(x), B(y)); \forall (x,y) \in X \times Y$$

Definition 1.1.8 [5, 37]

The product mapping $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ of the mapping $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ is defined by

$$(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2)),\ \forall (x_1, x_2) \in X_1 \times X_2$$

Lemma 1.1.9. [5]

Let $f_i : X_i \to Y_i$, $i = 1, 2$ are mapping and $A_i \in I^X$, $B_i \in I^Y$, $i = 1, 2$, we have

1. $(f_1 \times f_2)(A_1 \times A_2) \subset f_1(A_1) \times f_2(A_2)$.

2. $(f_1 \times f_2)^{-1}(B_1 \times B_2) = f_1^{-1}(B_1) \times f_2^{-1}(B_2)$

Definition 1.1.10 [30]

Let R_1 and R_2 be two fuzzy relations from X to Y, we may consider:

1. The fuzzy union of R_1 and R_2 denoted $R_1 \cup R_2$:

$$R_1 \cup R_2(x,y) = \max \{R_1(x,y), R_2(x,y)\} \quad \forall (x,y) \in X \times Y$$

2. The fuzzy intersection of R_1 and R_2 denoted $R_1 \cap R_2$:

$$R_1 \cap R_2(x,y) = \min \{R_1(x,y), R_2(x,y)\} \quad \forall (x,y) \in X \times Y$$

3. $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$

4. $(R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$ where $R_1^{-1}(x,y) = R_1(y,x)$

1.2. Proximity Uniformity And Syntopogenous structures.

Definition 1.2.1. [38]

A binary relation δ on the power set of a set X is called a proximity on X if δ satisfies the following axioms

P_1) $A\delta B$ implies $B\delta A$

P_2) $(A \cup B)\delta C$ iff $A\delta C$ or $B\delta C$

P_3) $A\delta B$ implies $A \neq \phi$, $B \neq \phi$

P_4) $A\not\delta B$ implies that there exists a subset E of X such that $A\not\delta E$ and $(X\text{-}E)\not\delta B$.

P_5) $A \cap B \neq \phi$ implies $A\delta B$

The pair (X, δ) is called a proximity space.

Theorm 1.2.2. [38]

If a subset A of a proximity space (X, δ) is defined to be closed iff $x\delta A$ implies $x \in A$, then the collection of complements of all closed sets so defined yields a topology.

Definition 1.2.3. [38]

If δ_1 and δ_2 are two proximities in a set X, we define $\delta_1 > \delta_2$ iff $A\delta_1 B$ implies $A\delta_2 B$. The above is expressed by saying that δ_1 is finer than δ_2 or δ_2 is coarser than δ_1.

Definition 1.2.4. [38]

Let (X, δ_1) and (Y, δ_2) be two proximity spaces. A function $f: X \to Y$ is said to be a proximity continuity iff

$$A\delta_1 B \text{ implies } f(A)\, \delta_2\, f(B).$$

Equivalently, f is a proximity continuity iff

$$C\delta_2 D \text{ implies } f^{-1}(C)\delta_1 f^{-1}(D)$$

Definition 1.2.5 [38]

A uniformity U on a set X is a collection of subsets of $X \times X$ satisfying the following conditions.

u_1) $\Delta \subset u$, $\forall u \in U$

u_2) $u, v \in U \Rightarrow u \cap v \in U$

u_3) $u \in U \Rightarrow$ there exists $v \in U$ s.t. $v \circ v \subset u$

u_4) $u \in U$, $u \subset v \subset X \times X \Rightarrow v \in U$

u_5) $u \in U \Rightarrow u^{-1} \in U$

The pair (X, U) is called a uniform space.

Definition 1.2.6 [38]

Let X be a non-empty set, $u \subset X \times X$, for $x \in X$ we define

$u[x] = \{y \in X / (x, y) \in U\}$, and $u[A] = \{y \in X \,|\, \exists\, x \in A,\ (x, y) \in u\}$

Definition 1.2.7 [38]

A topogenous order on a set X is a relation on the power set of X, denoted by $<$, satisfying :

i) $\phi < \phi$ and $X < X$

ii) $A < B$ implies $A \subset B$

iii) $A \subset C < D \subset B$ implies $A < B$

iv) $A < B$ and $C < D$ together imply $A \cap C < B \cap D$ and $A \cup C < B \cup D$

Definition 1.2.8. [38]

A syntopogenous structure on a set X is a family S of topogenous orders on X satisfying :

s_1) If $<_1$ and $<_2$ belong to S, there exists a $<$ in S such that $A < B$ whenever $A <_1 B$ or $A <_2 B$.

s_2) If $<$ belong to S, there exists a $<'$ in S such that $A < B$ implies the existence of a C satisfying $A <' C <' B$.

Example 1.2.9. [38]

Given a uniform space (X, U), one may define a proximity on X by $A \delta B$ iff every $u \in U$, one (and hence all) of the three following equivalent conditions is satisfied :

i) $U[A] \cap B \neq \phi$;

ii) $A \cap U[B] \neq \phi$;

iii) $(A \times B) \cap U \neq \phi$

where, $U[A] = \{y \in X | \exists x \in A \ \text{s.t.}\ (x, y) \in U\}$.

1.3. Multiple - Valued Logic.

We disply the fuzzy logical and corresponding set-theoretical notations used in this thesis. Let us denote by $[\alpha]$ the truth value of a proposition α when $[\alpha] \in I$. [33-35, 45]

1. $[\sim \alpha] := 1 - [\alpha]$
2. $[\alpha \wedge \beta] := \min\ ([\alpha], [\beta])$
3. $[\alpha \vee \beta] := \max\ ([\alpha], [\beta])$
4. $[\alpha \rightarrow \beta] := [\sim \alpha \vee \beta] = \min\ (1, 1 - [\alpha] + [\beta])$
5. $(\alpha \leftrightarrow \beta) := (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$
6. $(\alpha \vee \beta) = \sim(\sim \alpha \wedge \sim \beta), (\alpha \wedge \beta) = \sim(\sim \alpha \vee \sim \beta)$
7. $[\forall x \alpha(x)] := \inf_{x \in X} [\alpha(x)]$, $[\exists x \alpha(x)] = \sup_{x \in X} [\alpha(x)]$
8. $[\forall x \alpha(x)] = \sim [\exists x (\sim \alpha(x))]$

   where X is the universe of discourse.
9. $[x \in A] = A(x)$
10. $A \subset B := (\forall x)(x \in A \rightarrow x \in B)$
11. $A \equiv B := (A \subset B) \wedge (B \subset A)$

Lemma 1.3.1. [33]

Let $\{A_i \mid i \in I\}$ be an indexed family of a fuzzy sets in X, then :

1. $\models (\forall i)(i \in I \rightarrow A_i \subset B_i) \rightarrow \left(\bigcap_{i \in I} A_i \subset \bigcap_{i \in I} B_i \right)$
2. $\models (\forall i)(i \in I \rightarrow A_i \subset B_i) \rightarrow \left(\bigcup_{i \in I} A_i \subset \bigcup_{i \in I} B_i \right)$

3. $\models (\forall i)(i \in I \rightarrow A_i \equiv B_i) \rightarrow \left(\bigcap_{i \in I} A_i \equiv \bigcap_{i \in I} B_i \right)$

4. $\models (\forall i)(i \in I \rightarrow A_i \equiv B_i) \rightarrow \left(\bigcup_{i \in I} A_i \equiv \bigcup_{i \in I} B_i \right)$

Lemma 1.3.2. [35]

For any $A, B, C \in I^X$

1. $\models (A \subset B) \rightarrow ((B \subset C) \rightarrow (A \subset C))$

   and $\models (A \equiv B) \rightarrow ((B \equiv C) \rightarrow (A \equiv C))$

2. $\models (B \subset C) \rightarrow ((A \subset B) \rightarrow (A \subset C))$

   and $\models (B \equiv C) \rightarrow ((A \equiv B) \rightarrow (A \equiv C))$

Lemma 1.3.3. [35]

For any $A, B, C, D \in I^X$

1. $\models (A \subset B) \rightarrow (f(A) \subset f(B))$ and $\models (A \equiv B) \rightarrow (f(A) \equiv f(B))$

2. $\models (C \subset D) \rightarrow \left(f^{-1}(C) \subset f^{-1}(D)\right)$

   and $\models (C \equiv D) \rightarrow \left(f^{-1}(C) \equiv f^{-1}(D)\right)$

3. $\models (A \subset B) \rightarrow ((X - B) \subset (X - A))$

   and $\models (A \equiv B) \rightarrow ((X - A) \equiv (X - B))$

1.4. Fuzzifying Topological Spaces.

C.L. Chang introduced fuzzy set theory into topology [12], Wong, Lowen, Hotton, Pu and Liu, etc. discussed resp. various aspects of fuzzy topology [12, 17, 39]. In these authors'papers the fuzzy topology themselves remain almost as the classical topology for a family of open sets only some of these sets can be fuzzy.

Mingsheng Ying introduce a new approach for fuzzy topology [33-35] which depend on a many-valued logic. In this a new approach we find a real fuzzification of topology i.e. a fuzzy set may assume a partial openness and closedness.

1.4.a. Fuzzifying toplogical spaces.

Mingsheng Ying introduced the definition of fuzzifying topology [33] as,

Definition 1.4.a.1.

Let X be a universe of discourse, $\tau : 2^X \to I$ satisfy the following conditions:

1. $\tau(X) = 1$
2. $\forall A_1, A_2, \in 2^X, \tau(A_1 \cap A_2) \geq \tau(A_1) \wedge \tau(A_2)$
3. $\forall \{A_i : i \in I\} \subset 2^X, \tau\left(\bigcup_{i \in I} A_i\right) \geq \inf_{i \in I} \tau(A_i)$

Then τ is called a fuzzifying topology on X and (X, τ) is called a fuzzifying topological space.

Remark 1.4.a.2

The conditionds in Definition 1.4.a.1 may be rewritten respectively as follows :

i) $\models X \in \tau$

ii) $\forall A_1, A_2, \models (A_1 \in \tau) \wedge (A_2 \in \tau) \to A_1 \cap A_2 \in \tau$

iii) $\forall \{A_i : i \in I\} \subset 2^X, \models (\forall i)(i \in I \to A_i \in \tau) \to \bigcup_{i \in I} A_i \in \tau$

Specially,

If $\tau : 2^X \to \{0, 1\}$ then τ is a classical topology on X[28]; if $\tau : I^X \to \{0, 1\}$ then τ is a fuzzy topology on X[12]; and if $\tau : I^X \to I$ then τ is a smooth topology on X [42].

Example 1.4.a.2.

Let X be a non empty set. Define a mapping $\tau : 2^X \to I$ as :

$$\tau(A) = \begin{cases} 1 & ; \ A \in \{\phi, X\} \\ 0.5 & ; \ X \supset A \notin \{\phi, X\} \end{cases}$$

Clearly that τ is a fuzzifying topology but not classical topology.

Definition

If (X, τ) be a fuzzifying topology space and $Y \subset X$, then $\tau_Y : 2^Y \to I$ which is given as :

$$\tau_Y(v) = \sup_{V = U \cap Y} \tau(u)$$

is a fuzzifying topology.

Definition 1.4.a.3. [33]

The family of fuzzifying closed sets is denoted by $\Im$, and defined as follows :

$$A \in \Im := A^c \in \tau$$

i.e.

$$\Im(A) = \tau(A^c)$$

Theorm 1.4.a.4. [33]

Let $\Im: 2^X \to I$, where, $\Im(A) = \tau(A^c)$. Then $\Im$ satisfies the following conditions :

1. $\Im(\phi) = 1$
2. For any $A_1, A_2 \in 2^X$, $\Im(A_1 \cup A_2) \geq \Im(A_1) \wedge (A_2)$
3. For any $\{A_i : i \in I\} \subset 2^X$, $\Im\left(\bigcap_{i \in I} A_i\right) \geq \inf_{i \in I} \Im(A_i)$

Remark 1.4.a. 5.

A mapping $\Im: 2^X \to I$ which satisfies (1), (2) and (3) in theorem (1.4.a.4) is called a fuzzifying cotopology on X.

1.4.b. Fuzzifying neighborhood structure of a point.

Here, we build a fuzzifying neighborhood system and we give some of its properties.

Definition 1.4.b.1.[33]

Let (X, τ)be a fuzzfying topological space and $x \in X$. The neighborhood system of x is denoted by N_x and defined as, $N_x: 2^X \to I$ where,

$$N_x(A) = \sup_{x \in B \subset A} \tau(B).$$

Lemma 1.4.b.2.[33]

For any $A \in 2^X$, $\tau(A) = \inf_{x \in A} \sup_{x \in B \subset A} \tau(B)$

Theorm 1.4.b.3. [33]

For any $x \in X$ and $A \in 2^X$,

$$\tau(A) = \inf_{x \in A} \sup_{B \subset A} N_x(B)$$

Definition 1.4.b.3 [33]

A fuzzy set A is called normal, if $\sup_{x \in X} A(x) = 1$ and $(I^N)^{2^X}$ is the set of all normal fuzzy subsets of $2^X, \tau \subset 2^X$ is fuzzy normal if there exists $A \subset X, \tau(A) = 1$.

Theorm 1.4.b.4. [33]

The mapping $N : X \to \Im^N(2^X), x \to N_x$, has the following properties:

1. For any x, A $\models A \in N_x \to x \in A$
2. For any x, A, B $\models N_x(A \cap B) \geq N_x(A) \wedge N_x(B)$
3. For any x, A, B $\models A \subset B \Rightarrow N_x(A) \leq N_x(B)$
4. For any x, A,

$$\models A \in N_x \Rightarrow \exists C((C \in N_x) \wedge (C \subset A) \wedge \forall y(y \in C \Rightarrow C \in N_y))$$

Conversely, if a mapping N satisfies (2), (3), then τ is a fuzzifying topology which is defined as,

$$\tau(A) = \inf_{x \in A} N_x(A).$$

Specially, if it satisfies (1), (4) also, then for any $x \in X$, N_x is the neighborhood system of x w.r.t. τ.

1.4.c. Fundamental concepts.

Here we give the definitions of the derived set, closure, interior and boundary of a set in a fuzzifying topological space and we investigate some of there properties.

Derived set:

Definition 1.4.c.1. [24]

The fuzzifying derived set A' of A is defined as follows :

$$A'(x) = \inf_{B \cap (A - \{x\}) = \phi} \left(1 - N_x(B)\right)$$

Lemma 1.4.c.2. [33]

For any $x \in X$, $A \in 2^X$ we have

$$A'(x) = 1 - N_x\left(A^c \cup \{x\}\right)$$

Theorm 1.4.c.3. [33]

For any $A \in 2^X$, $\models A \in \Im \Leftrightarrow A' \subset A$

Proposition 1.4.c.4.

For any $A, B \in 2^X$,

1. $A \subset B \rightarrow A' \subset B'$
2. $(A \cup B)' = A' \cup B'$
3. $(A \cap B)' \subset A' \cap B'$
4. $(A \in \Im) \wedge (A' \subset A) \rightarrow A' \in \Im$

Proof.

1) $A'(x) = 1 - N_x\left(A^c \cup \{x\}\right) \le 1 - N_x\left(B^c \cup \{x\}\right) = B'\{x\}$

2) $(A \cup B)'(x) = 1 - N_x\left((A \cup B)^c \cup \{x\}\right)$

$$= 1 - N_x\left(\left(A^c \cap B^c\right) \cup \{x\}\right)$$
$$= 1 - N_x\left(\left(A^c \cup \{x\}\right) \cap \left(B^c \cup \{x\}\right)\right)$$
$$\le 1 - \min\left(N_x\left(A^c \cup \{x\}\right), N_x\left(B^c \cup \{x\}\right)\right)$$
$$= \max\left(1 - N_x\left(A^c \cup \{x\}\right), 1 - N_x\left(B^c \cup \{x\}\right)\right)$$
$$= \max\left(A'(x), B'(x)\right)$$
$$= \left(A' \cup B'\right)(x)$$

Then, $(A \cup B)' \subset (A' \cup B')$ (a)

In other side

$A \subset A \cup B$, $B \subset A \cup B \Rightarrow A' \subset (A \cup B)'$, $B' \subset (A \cap B)'$

$$\Rightarrow A' \cup B' \subset (A \cup B)' \qquad \text{(b)}$$

From (a) and (b) we have $(A \cup B)' = A' \cup B'$

3) $A \cap B \subset A$, $A \cap B \subset B \Rightarrow (A \cap B)' \subset A'$, $(A \cap B)' \subset B'$

$$\Rightarrow (A \cap B)' \subset A' \cap B'$$

4) $A \in \Im \Leftrightarrow A' \subset A$

$\Leftrightarrow A'' \subset A'$

$\Leftrightarrow A' \in \Im$

Closure:

Definition 1.4.c.5. [33]

Let (X, τ) be a fuzzifying topology space., $A \subset X$, the closure $\bar{A}$ of A is defined as,

$$\overline{A}(x) = \inf_{x \notin B \supset A} (1 - \Im(B))$$

Lemma 1.4.c.6. [33]

For any x, A, $\overline{A}(x) = 1 - N_x(A^c)$

Lemma 1.4.c.7.

For any x, A,

1) $\models \overline{A} = A \cup A'$

2) $\overline{A}(x) = \inf_{A \cap B = \phi} (1 - N_x(B))$

3) $\models A = \overline{A} \Leftrightarrow A \in \Im$

Proof.

1) If $x \in A$, $\quad (A \cup A')(x) = 1 \quad , \overline{A}(x) = 1 - N_x(A^c) = 1 - 0 = 1$

if $x \notin A$ $\quad (A \cup A')(x) = \max\{0, 1 - N_x(A^c \cup \{x\})\}$

$$= \max\{0, 1 - N_x(A^c)\}$$

$$= 1 - N_x(A^c)$$

$$= \overline{A}(x)$$

2) $\inf_{A \cap B = \phi} (1 - N_x(B)) = 1 - \sup_{A \cap B = \phi} N_x(B)$

$$= 1 - N_x(A^c)$$

$$= \overline{A}(x)$$

3) $\models A \in \Im \Leftrightarrow A' \subset A$

$$\Leftrightarrow A \cup A' = A$$

$$\Leftrightarrow \overline{A} = A.$$

Proposition 1.4.c.8.

For A, B $\in 2^X$, we have

1) $\overline{\phi} = \phi$

2) $A \subset \overline{A}$

3) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof.

1) For $x \in X$, $\overline{\phi}(x) = \inf\limits_{x \notin \phi \supset \phi} (1 - \tau(\phi)) = 0$ implies $\overline{\phi} = \phi$

2) since, $\overline{A} = A \cup A' \supset A \Rightarrow A \subset \overline{A}$

3) If $A \subset B$ then

$$\overline{A}(x) = 1 - N_x(A^c) \le 1 - N_x(B^c) = \overline{B}(x).$$

Now, $A \subset A \cup B,\ B \subset A \cup B \Rightarrow \overline{A} \cup \overline{B} \subset \overline{A \cup B}$

In the other hand,

$$\overline{(A \cup B)}(x) = \inf_{x \notin (A \cup B) \supset C} (1 - \tau(C))$$

$$\begin{aligned}
&= \inf_{x \notin (A \cup B) \supset (C_1 \cup C_2)} \left(1 - \tau(C_1 \cup C_2)\right) \\
&\le \inf_{x \notin (A \cup B) \supset (C_1 \cup C_2)} \left(1 - \min(\tau(C_1), \tau(C_2))\right) \\
&\le \inf_{x \notin (A \cup B) \supset (C_1 \cup C_2)} \max(1 - \tau(C_1),\ 1 - \tau(C_2)) \\
&= \inf_{x \notin A \subset C_1} \inf_{x \notin B \subset C_2} \max\left(1 - \tau(C_1),\ 1 - \tau(C_2)\right) \\
&\le \max\left(\inf_{x \notin A \subset C_1} (1 - \tau(C_1)),\ \inf_{x \notin B \subset C_2} (1 - \tau(C_2)) \right) \\
&= \overline{A}(x) \vee \overline{B}(x)
\end{aligned}$$

Hence, $\overline{(A \cup B)} \le (\overline{A} \cup \overline{B})(x) \Rightarrow \overline{A \cup B} \subset \overline{A} \cup \overline{B}$

Then, $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Definition 1.4.c.9. [33]

Let (X, τ) be a fuzzifying topological space, the interior of $A \in 2^X$ is defined as:

$$A^{o}(x) = N_x(A)$$

Theorm 1.4.c.10. [33]

For any x, A, B $\in 2^X$

1) $\models B \in \tau$ and $B \subset A \Rightarrow B \subset A^{o}$

2) $\models A \equiv A^{o} \Leftrightarrow A \in \tau$

3) $x \in A^{o} \Leftrightarrow (x \in A)$ and $\left(x \notin (X - A)^{\prime}\right)$

4) $A^{o} = X - \left(\overline{X - A}\right)$

Definition 1.4.c.11. [33]

Let (X, τ) be a fuzzifying topologicl space and $A \subset X$, the boundary of A is given as follows :

$$b(A)(x) = \min\left\{1 - A^{o}(x),\ 1 - (X - A)^{o}(x)\right\}$$

Theorm 1.4.c.12. [33]

For any A,

1) $b(A) = \overline{A} \cap \left(\overline{X - A}\right)$, and, $b(A) = b(X - A)$

2) $X - b(A) = A^{o} \cup (X - A)^{o}$

3) $\overline{A} = A \cup b(A)$, and, $b(A) \subset A \Leftrightarrow A \in \Im$

4) $A^{o} = A \cap (X - b(A))$, and, $b(A) \cap A \equiv \phi \Leftrightarrow A \in \tau$

Proposition 1.4.c.13.

Let (X, τ) be a fuzzifying topology space, for any A, B in X,

1) $A \subset B \Rightarrow A^{\circ} \subset B^{\circ}$

2) $(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}$

3) $(A \cup B)^{\circ} \supset A^{\circ} \cup B^{\circ}$

Proof.

1) $A \subset B \Rightarrow N_x(A) \le N_x(B)$ for all $x \in X$

$\Rightarrow A^{\circ}(x) \le B^{\circ}(x)$ for all $x \in X$

$\Rightarrow A^{\circ} \subset B^{\circ}$

2) $A \cap B \subset A,\ A \cap B \subset B \Rightarrow (A \cap B)^{\circ} \subset A^{\circ} \cap B^{\circ}$ (1

In the other hand,

$$\begin{aligned}(A \cap B)^{\circ}(x) &= N_x(A \cap B)\\ &\ge N_x(A) \wedge N_x(B)\\ &= A^{\circ}(x) \wedge B^{\circ}(x)\\ &= \left(A^{\circ} \cap B^{\circ}\right)(x)\end{aligned}$$

Hence, $(A \cap B)^{\circ} \supset A^{\circ} \cap B^{\circ}$ (2)

From (1), (2) we get $(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}$

3) Clearly by (1)

1.4.d. continuous functinos .

Definition 1.4.d.1. [35]

Let (X,τ), (Y, U) be two fuzzifying topological spaces. Aunary fuzzy predicate $C \in \Im\left(Y^X\right)$, called fuzzy continunity is given as follows :

$$C(f) := (\forall u)\left((u \in U) \to \left(f^{-1}(u) \in \tau\right)\right).$$

Intuitively, the degree to which f is continuous is

$$[C(f)] = \inf_{u \in 2^Y} \min\left(1, 1 - U(u) + \tau\left(f^{-1}(u)\right)\right).$$

Lemma 1.4.d.2. [35]

Let (X, τ), (Y, U), (Z, V) be three fuzzifying topological spaces. For any $f \in Y^X$, $g \in Z^Y$,

1) $\models C(f) \to (C(g) \to C(gof))$

2) $\models C(g) \to (C(f) \to C(fog))$

Lemma 1.4.d.3. [35]

Let (X, τ), (Y, U) be two fuzzifying topological spaces, $A \subset X$, for any $f \in Y^X$,

$$\models C(f) \to C(f/A)$$

1.4.e. The cuts and the representation of a fuzzifying topological space.

Now we study the cuts of a fuzzifying topology τ. Let us denote

$$\tau_\alpha = \left\{A \in 2^X \mid \tau(A) > \alpha\right\}$$

be the strong α - cut of τ

Lemma 1.4.e.1.

Let (X, τ) be a fuzzifying topological space, then for any α, τ_α is a topological spaces. Morever $\alpha_1 \le \alpha_2$ implies $\tau_{\alpha_1} \supset \tau_{\alpha_2}$.

Proof.

1) Since $\tau(X)=\tau(\phi)=1$, hence $\tau(X)=\tau(\phi)>\alpha$ for any α, then $X, \phi \in \tau_\alpha$

2) If A, $B \in \tau_\alpha$, then $\tau(A)>\alpha$, $\tau(B)>\alpha$, and so, $\tau(A)\wedge\tau(B)>\alpha$. Since $\tau(A\cap B)\ge\tau(A)\wedge\tau(B)$, then $\tau(A\cap B)>\alpha$, and $A\cap B\in\tau_\alpha$.

3) Consider $\{A_i \mid i\in I\}$ as a subfamily of τ_α, so $\tau(A_i)>\alpha$ for any $i\in I$. Since $\tau\left(\bigcup_{i\in I} A_i\right)\ge\inf_{i\in I}\tau(A_i)$, then $\tau\left(\bigcup_{i\in I} A_i\right)>\alpha$, and $\bigcup_{i\in I} A_i\in\tau_\alpha$.

Now, $A\in\tau_{\alpha_2} \Rightarrow \tau(A)>\alpha_2 \Rightarrow \tau(A)>\alpha_1 \Rightarrow A\in\tau_{\alpha_1}$

On the contrary, for the representation Theorm, we would like to see that if axioms for a fuzzifying topology are effectively satisfied when we start from a family of topologies associated with every α in I. More precisely.

Lemma 1.4.e.2.

Let $\{\tau_\alpha \mid \alpha\in I\}$ be a family of topologies on X such that $\alpha_1\ge\alpha_2$ implies $\tau_{\alpha_1}\subset\tau_{\alpha_2}$. Let τ be the fuzzy set of 2^X built by:

$$\tau(A)=\sup\{\alpha : A\in\tau_\alpha\},$$

then τ is a fuzzifying topology.

Proof.

1) Since $X, \phi\in\tau_1$, then $\tau(X)=\tau(\phi)=1$

2) For any $A, B\in 2^X$.

$$\begin{aligned}\tau(A\cap B) &= \sup\{\alpha \mid A\cap B\in\tau_\alpha\}\\ &\ge \sup\{\alpha \mid A\in\tau_\alpha, B\in\tau_\alpha\}\\ &= \tau(A)\wedge\tau(B)\end{aligned}$$

3) For a subfamily $\{A_i \mid i \in I\}$ of 2^X we have,

$$\tau\left(\bigcup_{i \in I} A_i\right) = \sup\left\{\alpha \,\middle|\, \bigcup_{i \in I} A_i \in \tau_\alpha\right\}$$

$$\geq \sup\{\alpha \mid A_i \in \tau_\alpha, \text{ for any } i \in I\}$$

$$= \inf_{i \in I} \sup\{\alpha \mid A_i \in \tau_\alpha\}$$

$$= \inf_{i \in I} \tau(A_i)$$

Theorm 1.4.e.3.

Let τ be a fuzzifying topology and let τ_α be the α-cut as defined precedingly. From the families of a topology τ_α build τ^* by.

$$\tau^*(A) = \sup\{\alpha \mid A \in \tau_\alpha\}.$$

Then $\tau = \tau^*$

Proof.

The proof is trivial from the preceding results and the well known fact that,

$$\sup\{\alpha \mid A \in \tau_\alpha\} = \sup\{\alpha \mid \tau(A) > \alpha\} = \tau(A).$$

Corollary 1.4.e.4.

τ is a fuzzifying topology iff for any $\alpha \in I$, τ_α is a classical topology.

Chapter II
Fuzzifying Uniform Spaces

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Fuzzifying Uniform Spaces

A uniformity is an important concept close to topology and a good tool for an investigation of topology. We must point out that Mingsheng Ying [36] was introduced a fuzzifying uniform spaces in the framework of a fuzzifying topology. Here we study the basic concept of fuzzifying uniform spaces and we establish some of their fundamental properties.

2.1. Fuzzifying uniform spaces

A fuzzifying uniform space is defined in [36] as:

Definition 2.1.1.

A fuzzifying uniformity on a set X is a function $U: 2^{X \times X} \to I^N$, "where, $(I^N)^{2^{X \times X}}$ is the set of all normal fuzzy subset of $2^{X \times X}$" which satisfies, for any u, v $\in 2^{X \times X}$,

fu$_1$) $\Delta \not\subset u \to U(u) = 0$,

fu$_2$) $U(u) \le U(u^{-1})$,

fu$_3$) $U(u) \le \sup\limits_{v \circ v \subset u} U(v)$,

fu$_4$) $U(u \cap v) \ge U(u) \wedge U(v)$,

fu$_5$) $u \subset v \to U(u) \le U(v)$.

The pair (X, U) is said to be a fuzzifying uniform space.

One may notice that the condition (fu$_1$) can be rewritten as :

$$U(u) = 1 \to \Delta \subset u.$$

Remark 2.1.2.

In a fuzzifying uniform space, we have $U(X \times X) = 1$, since from definition (2.1.1), there exist $u_o \subset X \times X$ s·t $U(u_o) = 1$ but, $U(u_o) \leq U(X \times X)$, then $U(X \times X) = 1$.

Remark 2.1.3.

Let (X, U) be a uniform space, this uniform can be identified with a fuzzifying uniformity U^*, $U^*:2^{X \times X} \to I$, where $U^*(u) = 1$ if $u \in U$ and $U^*(u) = 0$ if $u \notin U$.

Definition 2.1.4.

let (X, U) be a fuzzifying uniform space, $B:2^{X \times X} \to I^N$, and $B \subset U$. Then B, is a base for U if : $\forall u \in 2^{X \times X}$, $U(u) \leq \sup_{v \subset u} B(v)$.

Theorem 2.1.5.

Let $B:2^{X \times X} \to I^N$. Then B is a base for some fuzzifying uniformity on X iff for any u, $v \in 2^{X \times X}$,

a) $\Delta \not\subset u \to B(u) = 0$,

b) $B(u) \leq \sup_{v \subset u^{-1}} B(v)$,

c) $B(u) \leq \sup_{v \circ v \subset u} B(v)$,

d) $B(u) \wedge B(v) \leq \sup_{w \in u \wedge v} B(w)$.

Proof.

First: for any B satisfying a), b), c), and d) we set,

$$U(u) = \sup_{v \subset u} B(v)$$

and we prove that U is a fuzzifying uniformity as follow:

1) $(\Delta \not\subset u) \Rightarrow (\forall v)((v \subset u) \rightarrow \Delta \not\subset v)$

$$\Rightarrow B(v) = 0$$

$$\Rightarrow \sup_{v \subset u} B(v) = 0$$

$$\Rightarrow U(u) = 0 .$$

2) $U(u) = \sup_{v \subset u} B(v) \leq \sup_{v \subset u} \sup_{w \subset v^{-1}} B(w)$

$$= \sup_{w \subset v^{-1} \subset u^{-1}} B(w)$$

$$\leq \sup_{w \subset u^{-1}} B(w) = U(u^{-1}) .$$

3) $U(u) = \sup_{v \subset u} B(v) \leq \sup_{v \subset u} (\sup_{w \circ w \subset v} B(w))$

$$= \sup_{w \circ w \subset v \subset u} B(w)$$

$$\leq \sup_{w \circ w \subset u} B(w)$$

$$\leq \sup_{w \circ w \subset u} U(w) .$$

4) $U(u) \wedge U(v) \leq \sup_{w \subset u} B(w) \wedge \sup_{w_1 \subset v} B(w_1)$

$$= \sup_{w \subset u,\, w_1 \subset v} (B(w) \wedge B(w_1))$$

$$\leq \sup_{w \subset u,\, w_1 \subset v} (\sup_{w_o \subset w_1 \wedge w} B(w_o))$$

$$= \sup_{w_o \subset w_1 \cap w_1 \subset u \cap v} B(w_o)$$

$$\leq \sup_{w_o \subset u \wedge v} B(w_o) = U(u \wedge v) .$$

5) $U(u) = \sup_{w \subset u} B(w) \leq \sup_{w \subset v} B(w) = U(v)$ "Since, $u \subset v$".

Conversely now, we assume that U is a fuzzifying uniform space and B is a base for it, and we prove that B satisfies a), b), c), and d) as follows :

B_1) $\Delta \not\subset u \Rightarrow U(u) = 0$

$$\Rightarrow U(u) = 0 \text{ and } B \subset U$$
$$\Rightarrow B(u) = 0.$$

B_2) $B(u) \leq U(u)$

$$\leq U(u^{-1})$$
$$\leq \sup_{v \subset u^{-1}} B(v).$$

B_3) $B(u) \leq U(u) \leq \sup_{v \circ v \subset u} U(v)$

$$\leq \sup_{v \circ v \subset u} \sup_{w \subset v} B(w)$$
$$\leq \sup_{w \circ w \subset v \circ v \subset u} B(w)$$
$$\leq \sup_{w \circ w \subset u} B(w).$$

B_4) $B(u) \wedge B(v) \leq U(u) \wedge U(v)$

$$\leq U(u \cap v)$$
$$= \sup_{w \subset u \wedge v} B(w)$$

Definition 2.1.6.

Let (X, U) be a fuzzifying uniform space, $\varphi : 2^{X \times X} \to I^N$, and $\varphi \subset u$ if $\varphi^{\tilde{\cap}}$ is a base for U, then φ is called a subbase of U.

where,

$$\varphi^{\widetilde{\cap}}(u) = \sup\left\{ \bigwedge_{i=1}^{n} \varphi(u_i) \,\middle|\, \bigcap_{i=1}^{n} u_i = u,\ n \in N \right\}$$

Theorm 2.1.7.

If $\varphi : 2^{X \times X} \to I^N$ fulfils the following conditions for any u, $v \in 2^{X \times X}$

$s_1)$ $\Delta \not\subset u \to \varphi(u) = 0$

$s_2)$ $\varphi(u) \leq \sup_{v \subset u^{-1}} \varphi(v)$

$s_3)$ $\varphi(u) \leq \sup_{v \circ v \subset u} \varphi(v)$

Then φ is a subbase of some fuzzifying uniformity on X.

Proof.

We prove only that $\varphi^{\widetilde{\cap}}$ satisfies the conditions B_1)-B_4) as follows:

B_1) $\Delta \not\subset u \Rightarrow \Delta \not\subset \bigcap_{i=1}^{n} u_i$

$$\Rightarrow \Delta \not\subset u_{i_o}, \qquad i_o \in \{1, 2, \ldots, n\}$$

$$\Rightarrow \varphi(u_{i_o}) = 0, \qquad i_o \in \{1, 2, \ldots, n\}$$

$$\Rightarrow \bigwedge_{i=1}^{n} \varphi(u_i) = 0$$

$$\Rightarrow \quad \sup\left\{ \bigwedge_{i=1}^{n} \varphi(u_i) \,\middle|\, u = \bigcap_{i=1}^{n} u_i,\ n \in N \right\} = \varphi^{\widetilde{\cap}}(u) = 0$$

B_2) $\varphi^{\widetilde{\cap}} = \sup\left\{ \bigwedge_{i=1}^{n} \varphi(u_i) \,\middle|\, \bigcap_{i=1}^{n} u_i = u,\ n \in N \right\}$

$$\leq \sup\left\{ \bigwedge_{i=1}^{n} \sup_{v_i \subset u_i^{-1}} \varphi(v_i) \,\middle|\, \bigcap_{i=1}^{n} u_i = u,\ n \in N \right\}$$

$$= \sup\left\{ \sup_{f \in \prod_{i=1}^{n} M_i} \bigwedge_{i=1}^{n} \varphi(f(i)) \;\middle|\; \bigcap_{i=1}^{n} u_i = u,\ n \in N \right\}$$

where, $f \in \prod_{i=1}^{n} M_i := f : \{1, 2, \ldots, n\} \to 2^{X \times X}, \quad i \to v_i, \quad v_i \in M_i$ and

$$M_i := \{v_i : v_i \subset u_i^{-1}\}, \qquad i \in \{1, 2, \ldots, n\}$$

If $f \in \prod_{i=1}^{n} M_i$ then $\bigcap_{i=1}^{n} f(i) = \bigcap_{i=1}^{n} v_i \subset \bigcap_{i=1}^{n} u_i^{-1} = \left(\bigcap_{i=1}^{n} u_i\right)^{-1} = u^{-1}$

therefore,

$$\varphi^{\tilde{\frown}}(u) \le \sup\left\{ \bigwedge_{i=1}^{n} \varphi(v_i) \;\middle|\; v = \bigcap_{i=1}^{n} v_i \subset u^{-1}, \quad n \in N \right\}$$

$$= \sup_{v \subset u^{-1}} \varphi^{\tilde{\frown}}(v)$$

B$_3$) $\varphi^{\tilde{\frown}}(u) = \sup\left\{ \bigwedge_{i=1}^{n} \varphi(u_i) \;\middle|\; \bigcap_{i=1}^{n} u_i = U, \quad n \in N \right\}$

$$\le \sup\left\{ \bigwedge_{i=1}^{n} \sup_{v_i \circ v_i \subset u_i} \varphi(v_i) \;\middle|\; \bigcap_{i=1}^{n} u_i = u, \quad n \in N \right\}$$

$$= \sup\left\{ \sup_{f \in \prod_{i=1}^{n} M_i} \bigwedge_{i=1}^{n} \varphi(f(i)) \;\middle|\; \bigcap_{i=1}^{n} u_i \subset u, \quad n \in N \right\}$$

where, $f \in \prod_{i=1}^{n} M_i := f : \{1, 2, \ldots, n\} \to 2^{X \times X}, \quad i \to v_i, \quad v_i \in M_i$ and

$M_i := \{v_i : v_i \circ v_i \subset u_i\}, \qquad i = \overline{1, n}$

If $f \in \prod_{i=1}^{n} M_i$ then, $\bigcap_{i=1}^{n} v_i \circ \bigcap_{i=1}^{n} v_i \subset \bigcap_{i=1}^{n} (v_i \circ v_i) \subset \bigcap_{i=1}^{n} (u_i)$, therefore:

$$\widetilde{\varphi^{\cap}}(u) \le \sup\left\{ \bigwedge_{i=1}^{n} \varphi(v_i) \,\middle|\, v \circ v = \bigcap_{i=1}^{n} v_i \circ \bigcap_{i=1}^{n} v_i \subset \bigcap_{i=1}^{n}(u_i) = u \right\}$$

$$= \sup_{v \circ v \subset u} \widetilde{\varphi^{\cap}}(v)$$

$$B_{4}) \ \widetilde{\varphi^{\cap}}(u) \wedge \widetilde{\varphi^{\cap}}(v) = \sup\left\{ \bigwedge_{i=1}^{n} \varphi(u_i) \,\middle|\, \bigcap_{i=1}^{n} u_i = u, \quad n \in N \right\}$$

$$\wedge \sup\left\{ \bigwedge_{i=1}^{n} \varphi(v_i) \,\middle|\, \bigcap_{i=1}^{n} v_i = v, \quad n \in N \right\}$$

$$= \sup\left\{ \bigwedge_{i=1}^{n} \phi(u_i) \wedge \bigwedge_{i=1}^{n} \phi(v_i) \,\middle|\, \bigcap_{i=1}^{n} v_i = v, \quad \bigcap_{i=1}^{n} u_i = u, \quad n \in N \right\}$$

$$\le \sup\left\{ \bigwedge_{i=1}^{n} \varphi(w_i) \,\middle|\, \bigcap_{i=1}^{n}(w_i) \subset u \cap v, \quad n \in N \right\}$$

$$= \sup_{w \subset u \cap v} \widetilde{\varphi^{\cap}}(w)$$

Remark 2.1.8.

From Definition (2.1.4.), clearly every uniform space is a base of it self. And so, every uniform space satisfies directly the conditions of Theorem (2.1.5.), consequently we take the following Theorem.

Theorem 2.1.9.

If U_i is a fuzzifying uniformity on X, for any $i \in I$, then $\bigcup_{i \in I} U_i$ is a subbase of some fuzzifying uniformity on X .

Proof.

For $u \in 2^{X \times X}$ we define $(\bigcup_{i \in I} U_i)(u) = \sup_{i \in I} U_i(u)$ and we prove that $\bigcup_{i \in I} U_i$ verfiy the conditions s_1), s_2) and s_3) as follows:

$s_1)\ \Delta \not\subset u \Rightarrow U_i(u) = 0 \quad$ for any $i \in I$

$$\Rightarrow \sup_{i \in I} U_i(u) = 0$$

$$\Rightarrow \left(\bigcup_{i \in I} U_i \right)(u) = 0$$

$$s_2)\ \left(\bigcup_{i \in I} U_i \right)(u) = \sup_{i \in I} U_i(u)$$

$$\leq \sup_{i \in I} \sup_{v \subset u^{-1}} U_i(v)$$

$$= \sup_{v \subset u^{-1}} \left(\bigcup_{i \in I} U_i \right)(v)$$

$$s_3)\ \left(\bigcup_{i \in I} U_i \right)(u) = \sup_{i \in I} U_i(u)$$

$$\leq \sup_{i \in I} \sup_{v \circ v \subset u} U_i(v)$$

$$= \sup_{v \circ v \subset u} \left(\bigcup_{i \in I} U_i \right)(v).$$

2.2. Fuzzifying uniform topology

Here we build a fuzzifying topology from a fuzzifying uniformity and we give some of its properties.

Lemma 2.2.1.

Let (X, U) be a fuzzifying uniform space, and $\tau : 2^X \to I$ defined by,

$$A \in \tau := (\forall x)((x \in A) \to (\exists u)((u \in U) \wedge (U[x] \subseteq A))), \quad A \subset X$$

i.e.

$$\tau(A) = \inf_{x \in A} \sup_{u[x] \subseteq A} U(u), \qquad A \subset X$$

Then τ is a fuzzifying topology on X and called fuzzifying uniform topology of U.

Proof.

o_1) Clearly, for any $x \in X$, $X \times X\,[x] = X$, since, $U(X \times X) = 1$,

then $\tau(X) = 1$ (from definition 1.1.2)

o_2) For any $A_1, A_2 \subset X$ we have,

$$\tau(A_1) \wedge \tau(A_2) = \inf_{x_1 \in A_1} \sup_{U_1(x_1) \subset A_1} U(u_1) \wedge \inf_{x_2 \in A_2} \sup_{U_2(x_2) \subset A_2} U(u_2)$$

$$\leq \inf_{x \in A_1 \cap A_2} \Big(\sup_{U_1[x] \subset A_1} U(u_1) \wedge \sup_{U_2[x] \subset A_2} U(u_2) \Big)$$

$$= \inf_{x \in A_1 \cap A_2} \sup_{\substack{U_1[x] \subset A_1, \\ U_2[x] \subset A_2}} (U(u_1) \wedge U(u_2))$$

$$\leq \inf_{x \in A_1 \cap A_2} \sup_{\substack{u_1[x] \subset A_1, \\ u_2[x] \subset A_2}} U(u_1 \cap u_2)$$

$$\leq \inf_{x \in A_1 \cap A_2} \sup_{u[x] \subset A_1 \cap A_2} U(u)$$

because, $(u_1 \cap u_2)[x] = u_1[x] \cap u_2[x]$ to show that,

$$y \in (u_1 \cap u_2)[x] \Leftrightarrow (x, y) \in u_1 \cap u_2$$

$$\Leftrightarrow (x, y) \in u_1 \wedge (x, y) \in u_2$$

$$\Leftrightarrow y \in u_1[x] \wedge y \in u_2[x]$$

$$\Leftrightarrow y \in u_1[x] \cap u_2[x]$$

0_3) For any I, we have,

$$\tau\Big(\bigcup_{i \in I} A_i\Big) = \inf_{x \in \bigcup_{i \in I} A_i} \sup_{U[x] \subset \bigcup_{i \in I} A_i} U(u)$$

$$= \inf_{i \in I} \inf_{x \in A_i} \sup_{u[x] \subset \bigcup_{i \in I} A_i} U(u)$$

$$\geq \inf_{i \in I} \inf_{x \in A_i} \sup_{u[x] \subset A_i} U(u)$$

because, $u[x] \subset A_i \subset \bigcup_{i \in I} A_i$ and so, $\sup_{u[x] \subset A_i} U(u) \leq \sup_{u[x] \subset \bigcup_{i \in I} A_i} U(u)$.

Theorem 2.2.2. [36]

Let (X, U) be fuzzifying uniform space and τ the fuzzifying topology of U. Then for any $x \in X$, $A \subseteq X$,

$\models (x \in A^{\circ}) \leftrightarrow (\exists u)(u \in U) \wedge (u[x] \subseteq A))$.

Theorem 2.2.3 [36]

Let (X, U) be a fuzzifying uniform space and τ the fuzzifying topology of U then for any $A \subseteq X$, $x \in X$,

$\models x \in \overline{A} \leftrightarrow (\forall u)((u \in U) \rightarrow (x \in u[A]))$,

i.e.

$$\bar{A}(x) = \inf_{x \notin u[A]} (1\text{-}U(u)).$$

where $\overline{A}$ is the closure of A with respect to τ.

2.3. Fuzzy uniform continuity

Definition 2.3.1.

Let (X, U) and (Y, V) be two fuzzifying uniform spaces. A unary fuzzy predicate $\widetilde{C} \in \Im(Y^X)$, called fuzzy uniform continuity, is defined as follows:

$$\widetilde{C}(f) := (\forall v)(v \in V) \rightarrow ((f \times f)^{-1}(v) \in U)), \; f \in Y^X.$$

Intuitively, the degree to which f is continuous is,

$$[\widetilde{C}(f)] = \inf_{v \in 2^Y} \min(1, 1 - V(v) + U(f \times f)^{-1}(v)))$$

Lemma 2.3.2.

Let (X, U), (Y, V) and (Z, W) be three fuzzifying uniform spaces, for any $f \in Y^X$, $g \in Z^Y$,

(1) $\models \widetilde{C}(f) \to (\widetilde{C}(g) \to \widetilde{C}(gof))$.

(2) $\models \widetilde{C}(g) \to (C(f) \to C(fog))$.

Proof.

We only demonstrate (1). It suffices to show that

$[\widetilde{C}(f)] \leq [\widetilde{C}(g) \to \widetilde{C}(gof)]$

If $[\widetilde{C}(g)] \leq [\widetilde{C}(gof)]$, its obvious.

If $[\widetilde{C}(g)] > [\widetilde{C}(gof)]$, then

$$
\begin{aligned}
[\widetilde{C}(g)] - [\widetilde{C}(gof)] &= \inf_{w \in 2^Z} \min(1, 1 - W(w) + V((g \times g)^{-1}(w))) \\
&\quad - \inf_{w \in 2^Z} \min(1, 1 - W(w) + U(((gof) \times (gof))^{-1}(w)) \\
&\leq \sup_{w \in 2^Z} (V((g \times g)^{-1}(w)) - U((gof) \times (gof))^{-1}(w)) \\
&= \sup_{w \in 2^Z} (V(g \times g)^{-1}(w) - U(((g \times g) o (f \times f))^{-1}(w))) \\
&= \sup_{w \in 2^Z} (V(g \times g)^{-1}(w) - U((f \times f)^{-1}((g \times g)^{-1}(w)))) \\
&\leq \sup_{v \in 2^Y} (V(v) - U((f \times f)^{-1}(v)))
\end{aligned}
$$

Therefore

$$
\begin{aligned}
[\widetilde{C}(g) \to \widetilde{C}(gof)] &= \min(1, 1 - [\widetilde{C}(g)] + \widetilde{C}(gof)) \\
&\geq \inf_{v \in 2^Y} \min(1, 1 - V(v) + U((f \times f)^{-1}(v))) \\
&= [\widetilde{C}(f)]
\end{aligned}
$$

Proposition 2.3.3.

Let (X, U) and (Y, V) be two fuzzifying uniform spaces, and τ_1 and τ_2 be the fuzzifying topologies of U and V, respectively. For any $f \in Y^X$,

$$\models \tilde{C}(f) \to C(f).$$

where C is fuzzy continuity w. r. t. τ_1 and τ_2 (see, Definition 1.4.d.1)

Proof.

For any $x \in f^{-1}(B)$, if $v \in 2^{Y \times Y}$ with $v[f(x)] \subset B$, then we prove that $(f \times f)^{-1}(v)[x] \subset f^{-1}(B)$?

Since, $y \in (f \times f)^{-1}(v)[x] \Rightarrow (x, y) \in (f \times f)^{-1}(v)$

$$\Rightarrow (f \times f)(x, y) \in v$$

$$\Rightarrow (f(x), f(y)) \in v$$

$$\Rightarrow f(y) \in v[f(x)].$$

Since $f(y) \in v[f(x)]$ and $v[f(x)] \subset B$ then $f(y) \in B$ and so $y \in f^{-1}(B)$.

Hence, $\sup_{u[x] \subset f^{-1}(B)} U(u) \geq \sup_{v[f(x)] \subset B} U((f \times f)^{-1}(v))$. Also we have,

$$\inf_{x \in f^{-1}(B)} \sup_{u[x] \subset f^{-1}(B)} U(u) \geq \inf_{x \in f^{-1}(B)} \sup_{v[f(x)] \subset B} U((f \times f)^{-1}(v))$$

$$= \inf_{f(x) \in B} \sup_{v[f(x)] \subset B} U((f \times f)^{-1}(v))$$

$$\geq \inf_{y \in B} \sup_{v[y] \subset B} U((f \times f)^{-1}(v))$$

$$[C(f)] = \inf_{B \in 2^Y} \min(1, 1 - \tau_2(B) + \tau_1(f^{-1})(B)))$$

$$= \inf_{B \in 2^Y} \min(1, 1 - \inf_{y \in B} \sup_{v[y] \subset B} V(v) + \inf_{x \in f^{-1}(B)} \sup_{U[x] \subset f^{-1}(B)} U(u))$$

$$\geq \inf_{B \in 2^Y} \min(1, 1 - \inf_{y \in B} \sup_{V[y] \subset B} V(v) + \inf_{y \in B} \sup_{V[y] \subset B} U((f \times f)^{-1}(v)))$$

$$\geq \inf_{B\in 2^Y} \inf_{y\in B} \inf_{v[y]\subset B} \min(1, 1 - V(v) + U((f\times f)^{-1}(v)))$$

$$= [\tilde{C}(f)].$$

2.4.The cuts and the representation of a fuzzifying uniform space

To study the cuts of a fuzzifying uniformity U, Let us denote

$$U_\alpha = \{u \in 2^{X\times X} \backslash\ U(u) > \alpha\}$$

U_α is said to be α-cut of a fuzzifying uniformity U.

Definition 2.4.1

If U and V are uniformities (fuzzifying uniformities) on X s.t. $U \subset V$ ($\forall$ $u \in 2^{X\times X}$, $U(u) \leq V(v)$) then we say that V is finer than U or that U is coarser than V.

Lemma 2.4.2

Let (X, U) be a fuzzifying uniform space, then for any α in I, U_α is a uniformity on X. Moreover, if $\alpha_1 \geq \alpha_2$ then $U_{\alpha_1} \subset U_{\alpha_2}$.

Proof.

u_1) If $u\in U_\alpha$, $\alpha \geq 0$, then $U(u) > \alpha \geq 0$. Hence $\Delta \subset u$.

u_2) For $u \subset v$ and $u \in U_\alpha$, since $U(u) > \alpha$ and $U(u) \leq U(v)$ thus $U(v) > \alpha$, then $v \in U_\alpha$.

u_3) If $u \in U_\alpha$, since $U(u) \leq \sup_{v\circ v\subset u} U(v)$, then $\sup_{v\circ v\subset u} U(v) > \alpha$ thus there exist $v \in 2^{X\times X}$ s.t. $v\circ v \subset u$ and $U(v) > \alpha$, then $v \in U_\alpha$, $v\circ v \subset u$.

u_4) If u, v $\in U_\alpha$ then $U(u) \wedge U(v) > \alpha$, since $U(u \cap v) \geq U(u) \wedge U(v)$ thus $U(u \cap v) > \alpha$ then $u \cap v \in U_\alpha$.

u_5) If $u \in U_\alpha$, then $U(u) > \alpha$, since $U(u) \leq U(u^{-1})$ so that $U(u^{-1}) > \alpha$, then $u^{-1} \in U_\alpha$. Hence U_α is a uniformity on X. The second part is trivial to verify.

On the contrary, for the representation theorem, we would like to see that if the axioms for a fuzzifying uniformity are effectively satisfied when we start from a family of uniformity associated with every α in I

More precisely .

Lemma 2.4.3.

Let $\{ U_\alpha \backslash \alpha \in I\}$ be a family of uniformity on X such that $\alpha_1 \geq \alpha_2$ implies $U_{\alpha_1} \subset U_{\alpha_2}$ Let U be a fuzzy set built by .

$$U(u) = \sup \{\alpha \backslash u \in U_\alpha\}$$

Then U is a fuzzifying uniformity on X.

Proof.

fu_1) If $\Delta \not\subset u$, then $u \notin U_\alpha$ for any α in I, then $U(u) = 0$

fu_2) If $u \in U_\alpha$ and $u \subset v$, then $v \in U_\alpha$,

since $\{\alpha \backslash u \in U_\alpha\} \leq \{\alpha \backslash v \in U_\alpha\}$, then $U(u) \leq U(v)$.

fu_3) Since, $\sup \{\alpha \backslash u \in U_\alpha\} \leq \sup \{\alpha \backslash u^{-1} \in U_\alpha\}$,

then $U(u) \leq U(u^{-1})$.

fu_4) Since, $\sup \{\alpha \backslash u \cap v \in U_\alpha\} \geq \sup \{\alpha \backslash u \in U_\alpha$ and, $v \in U_\alpha\}$,

then $U(u \cap v) \geq U(u) \wedge U(v)$.

fu_5) $U(u) = \sup \{\alpha \backslash u \in U_\alpha\}$

$$\leq \sup \{\alpha \backslash (\exists v)(v \circ v \subset u \rightarrow v \in U_\alpha\}$$

$$= \sup_{v \circ v \subset u} \sup \{\alpha \backslash v \in U_\alpha\}$$

$$= \sup_{v \circ v \subset u} U(v)$$

Theorem 2.4.4.

Let U be a fuzzifying uniformity on X and U_α be the strong α-cut as defined precedingly. From the families of uniformity U_α on X we build U^* by $U^*(u) = \sup \{\alpha \backslash u \in U_\alpha\}$

Then $U^* = U$.

Proof.

The proof is trivial from the preceding results and the well known fact that, $\sup \{\alpha \backslash u \in U_\alpha\} = \sup \{\alpha \backslash U(u) > \alpha\} = U(u)$.

Chapter III
Fuzzifying Proximity Spaces

Chapter III

Fuzzifying Proximity Spaces

Recently, a number of authors have expanded a new approach to fuzzification. The classical approach was to maintain the classical mathimatical structures which alow the sets to be fuzzy. For instance, the way Chang [12] defined fuzzy topology is exactly the classic definition for a family of open sets, only the fact that some of these sets can be fuzzy makes the difference. In the new approach we reformulate the defining axioms themselves in terms of multiple valued logic. In this case the corresponding axioms become systems of inequalities. The basic idea is to see an axiom like $P \rightarrow Q$ as defining a constraint between the truth values p, q (of P, Q respectively). Here we take $q \geq p$. See for example Badard [11] and Ramadan [42]. Under the title " A new approach for fuzzy topology ", Mingsheng Ying introduced three papers [33-35] about the concept of fuzzifying topology, Also in [36] he introduced the concept of fuzzifying uniform spaces. In this chapter we introduce the concept of fuzzifying proximity spaces and we make some links between fuzzifying proximity spaces and fuzzifying topological spaces on the one hand and fuzzifying uniform spaces on the other hand.

3.1. Fuzzifying proximity spaces

Definition 3.1.1.

A fuzzifying proximity on a set X is a function $\delta \in I^{2^X \times 2^X}$ which satisfies.

fp_1) $\delta(A, B) = \delta(B, A)$

fp_2) $\delta(A \cup B, C) = \delta(A, C) \vee \delta(B, C)$

fp_3) $\delta(\phi, X) = 0$

fp_4) For any $A, B \in 2^X$ there exist $E \subset X$ S.t.

$$\delta(A, B) \geq \delta(A, E) \vee \delta(X - E, B)$$

fp5) $\delta(A, B) \geq (A \cap B)(x)$ for any $x \in X$

The pair (X, δ) is called a fuzzifying proximity space and the number $\delta(A, B)$ can be interpreted as the degree of nearness of the sets A and B.

Remark 3.1.2.

Let (X, δ) be a proximity space. This proximity can be identified with a fuzzifying δ^* where $\delta^*: 2^X \times 2^X \to I$ as:

$$\delta^*(A, B) = 1 \text{ if } (A, B) \in \delta;\ \delta^*(A, B) = 0 \text{ if } (A, B) \notin \delta.$$

Remark 3.1.3.

Let (X, δ) be a fuzzifying proximity space, then.

1) $A \subset C$, $B \subset D$ implies $\delta(A, B) \leq \delta(C, D)$

2) $\delta(X, X) = 1$.

Definition 3.1.4.

Let (X, δ_1) and (Y, δ_2) be two fuzzifying proximity spaces, A unary fuzzy predicate $\tilde{\tilde{C}} \in \Im(Y^X)$, called fuzzifying proximity continuity is defined as the following two equivalent conditions:

1) $\tilde{\tilde{C}}(f) := (\forall A, B)((A, B) \in \delta_1 \to (f(A), f(B)) \in \delta_2)$

2) $\tilde{\tilde{C}}(f) := (\forall C, B)((f^{-1}(C), f^{-1}(D)) \in \delta_1 \to (C, D) \in \delta_2)$

Intutively, the degree to which f is continuous is

$$[\tilde{\tilde{C}}(f)] = \inf_{A,B \in 2^X} \min(1, 1-\delta_1(A, B) + \delta_2(f(A), f(B)))$$

$$= \inf_{C,D \in 2^Y} \min(1, 1-\delta_1(f^{-1}(C), f^{-1}(D)) + \delta_2(C, D)).$$

Theorem 3.1.5.

let (X, δ_1), (Y, δ_2) and (Z, δ_3) be three fuzzifying proximity spaces. For any $f \in Y^X$, $g \in Z^Y$,

(1) $\models \tilde{\tilde{C}}(f) \to (\tilde{\tilde{C}}(g) \to \tilde{\tilde{C}}(gof))$

(2) $\models \tilde{\tilde{C}}(g) \to (\tilde{\tilde{C}}(f) \to \tilde{\tilde{C}}(fog))$.

Proof.

We only demonstrate (1). It suffices to show that

$[\tilde{\tilde{C}}(f)] \le [\tilde{\tilde{C}}(g) \to \tilde{\tilde{C}}(gof)]$.

If $[\tilde{\tilde{C}}(g)] \le [\tilde{\tilde{C}}(gof)]$, its obvious.

If $[\tilde{\tilde{C}}(g)] > [\tilde{\tilde{C}}(gof)]$, then

$$[\tilde{\tilde{C}}(g)] - [\tilde{\tilde{C}}(gof)] = \inf_{C,D \in 2^Z} \min(1, 1-\delta_1(g^{-1}(C), g^{-1}(D)) + \delta_3(C,D)) -$$

$$- \inf_{C,D \in 2^Z} \min(1, 1-\delta_2((gof)^{-1}(C), (gof)^{-1}(D)) + \delta_3(C,D)).$$

$$\leq \sup_{C, D \in 2^Z} (\delta_1(f^{-1}(g^{-1}(C)), (f^{-1}(g^{-1}(D))) - \delta_2(g^{-1}(C), g^{-1}(D)))$$

$$\leq \sup_{A, B \in 2^Y} (\delta_1(f^{-1}(A), f^{-1}(B)) - \delta_2(A, B))$$

Therefore,

$$[\tilde{\tilde{C}}(g) \to \tilde{\tilde{C}}(gof)] = \min(1, 1-[\tilde{\tilde{C}}(g)] + [\tilde{\tilde{C}}(gof)])$$

$$\geq \inf_{A, B \in 2^Y} \min(1, 1-\delta_1(f^{-1}(A), f^{-1}(B)) - \delta_2(A, B))$$

$$= [\tilde{\tilde{C}}(f)].$$

Definition 3.1.6.

Let (X, δ) be a fuzzifying proximity space, Y be a subset of X. For $A, B \in 2^Y$ we define, $\delta_Y(A, B) = \delta(A, B)$.

It is easily verified that δ_Y is a fuzzifying proximity on Y. We call δ_Y the subspase proximity.

Lemma 3.1.7.

Let (X, δ_1), (Y, δ_2) be two fuzzifying proximity spaces, $A \subset X$. for any $f \in Y^X$,

$$\models c(f) \to c(f/A).$$

Proof.

$$[c(f/A)] = \inf_{M, N \in 2^A} \min(1, 1-\delta_{1A}(M, N) + \delta_2(f/A(M), (f/A(N)))$$

$$= \inf_{M, N \in 2^A} \min(1, 1-\delta_1(M, N) + \delta_2(f(M), f(N)))$$

$$\geq \inf_{M, N \in 2^X} \min(1, 1-\delta_1(M, N) + \delta_2(f(M), f(N)))$$

$$= [c(f)].$$

3.2. The cuts and the representation of fuzzifying proximity spaces

Now we study the cuts of a fuzzifying proximity δ. Let us denote

$$\delta_\alpha = \{(A, B) \in 2^X \times 2^X \setminus \delta(A, B) > \alpha\}$$

δ_α is said to be strong α-cut of fuzzifying proximity δ.

Definition 3.2.1.

If δ_1 and δ_2 are proximities (Fuzzifying proximities) on X such that $\delta_1 \supset \delta_2$ ($\forall A, B \in 2^X \times 2^X$, $\delta_1(A, B) \leq \delta_2(A, B)$), then we say that δ_1 is finer than δ_2 or δ_2 is coarser than δ_1.

Lemma 3.2.2.

Let (X, δ) be a fuzzifying proximity space, then for any α in I, δ_α is a proximity on X. Moreover, if $\alpha_1 \geq \alpha_2$ then $\delta_{\alpha_1} \subset \delta_{\alpha_2}$.

Proof.

p_1) $(A, B) \in \delta_\alpha \Leftrightarrow \delta(A, B) > \alpha$

$\Leftrightarrow \delta(B, A) > \alpha$

$\Leftrightarrow (B, A) \in \delta_\alpha$.

p_2) $(A \cup B, C) \in \delta_\alpha \Leftrightarrow \delta(A \cup B, C) > \alpha$

$\Leftrightarrow \delta(A, C) \vee \delta(B, C) > \alpha$

$\Leftrightarrow \delta(A, C) > \alpha$ or $\delta(B, C) > \alpha$

$\Leftrightarrow (A, C) \in \delta_\alpha$ or $(B, C) \in \delta_\alpha$.

p_3) Sinc $\delta(\phi, X) = 0$ then $(\phi, X) \notin \delta_\alpha$ for any α in I.

p_4) $(A, B) \notin \delta_\alpha \Rightarrow \delta(A, B) \leq \alpha$

$\Rightarrow$ ther exist $E \subset X$ s.t. $\delta(A, E) \vee \delta(X\text{-}E, B) \leq \alpha$

$\Rightarrow$ ther exist $E \subset X$ s.t. $\delta(A, E) \leq \alpha$ and $\delta(X\text{-}E, B) \leq \alpha$

$\Rightarrow$ ther exist $E \subset X$ s.t. $(A, E) \notin \delta_\alpha$ and $(X\text{-}E, B) \notin \delta_\alpha$

p_5) $A \cap B \neq \phi \Rightarrow (A \cap B)(x) \neq 0$ for some x

$\Rightarrow \delta(A, B) = 1$

$\Rightarrow \delta(A, B) > \alpha$ for any α

$\Rightarrow (A, B) \in \delta_\alpha$.

Henc δ_α is a proximity on X. The second part is trivial to verify.

On the contrary, for the representation theorem, we would like to see that if the axioms for a fuzzifying proximity are effectively satisfied when we start from a family of proximity associated with every α in I.

More precisely.

Lemma 3.2.3.

Let $\{\delta_\alpha \backslash \alpha \in I\}$ be a family of proximity on X such that $\alpha_1 \geq \alpha_2$ implies $\delta_{\alpha_1} \subset \delta_{\alpha_2}$. Let δ be a fuzzy set built by.

$$\delta(A, B) = \sup \{\alpha \backslash (A, B) \in \delta_\alpha\}$$

Then δ is a fuzzifying proximity on X .

Proof.

fp_1) $\delta(A, B) = \sup \{\alpha \backslash (A, B) \in \delta_\alpha\}$

$\leq \sup \{\alpha \backslash (B, A) \in \delta_\alpha\}$

$= \delta(B, A)$.

Similarly, $\delta(B, A) \leq \delta(A,B)$. Then $\delta(A, B) = \delta(B, A)$.

fp_2) $\delta(A \cup B, C) = \sup\{\alpha \backslash (A \cup B, C) \in \delta_\alpha\}$

$= \sup \{\alpha \backslash (A, C) \in \delta_\alpha \text{ or } (B, C) \in \delta_\alpha\}$

$= \sup \{\alpha \backslash (A, C) \in \delta_\alpha\} \vee \{\alpha \backslash (B, C) \in \delta_\alpha\}$

$= \delta(A, C) \vee \delta(B, C)$

fp_3) Since $(\phi, X) \notin \delta_\alpha\ \forall \alpha \in I$ then $\delta(\phi, X) = 0$.

fp_4) For any A, B $\in 2^X$ there exist $E \subset X$ s.t.

$\delta(A, E) \vee \delta(X\text{-}E, B) = \sup \{\alpha \backslash (A, E) \in \delta_\alpha\} \vee \sup\{\beta \backslash (X\text{-}E, B) \in \delta_\beta\}$

$= \sup\{\gamma \backslash (A, E) \in \delta_\gamma \text{ or} (X\text{-}E, B) \in \delta_\gamma, \gamma = \alpha \vee \beta\}$

$\leq \sup \{\gamma \backslash (A, B) \in \delta_\gamma\}$

$= \delta(A, B)$.

fp_5) If $A \cap B = \phi$, then $(A \cap B)(x) \leq \delta(A, B)$,

If $A \cap B \neq \phi$ implies $(A, B) \in \delta_\alpha\ \forall \alpha \in I$ then $\delta(A, B) = 1$.

Therefore, $\delta(A, B) \geq (A \cap B)(x)\ \forall x \in X$.

Theorem 3.2.4.

Let δ be a fuzzifying proximity on X and δ_α be the α-cut as defined precedingly. From the families of proximity δ_α on X we build δ^* by

$$\delta^*(A, B) = \sup \{\alpha \backslash (A, B) \in \delta_\alpha\}$$

Then $\delta^* = \delta$.

Proof.

The proof is trivial from the preceding results and the well known fact that, $\sup \{\alpha \backslash (A, B) \in \delta_\alpha\} = \sup \{\alpha \backslash \delta(A, B) > \alpha\} = \delta (A, B)$

3.3. Fuzzifying topology induced by a fuzzifying proximity

Remark 3.3.1.

A fuzzifying cotopology on X is a map $\Im: 2^X \to I$ where $\Im(A)=\tau(A^c)$ which satisfies

a) $\Im(\phi) = 1$;

b) For any $A_1, A_2 \in 2^X$, $\Im(A_1 \cup A_2) \geq \Im(A_1) \wedge \Im(A_2)$;

c) For any $\{A_i \backslash i \in I\}$, $\Im(\bigcap_{i\in I} A_i) \geq \inf_{i\in I} \Im(A_i)$.

Theorem 3.3.3.

Let δ be a fuzzifying proximity on X and $\Im: 2^X \to I$ defined by

$$A \in \Im := (\forall x)((\{x\}, A) \in \delta \to x \in A)$$

i.e.

$$\Im(A) = \inf_{x \in A^c} (1 - \delta(\{x\}, A^c)$$

then $\Im$ is a fuzzifying cotopology on X.

Proof.

a) Sinc $\delta(\{x\}, \phi) = 0$ then $\Im(\phi) = 1$

b) For any $A_1, A_2 \in 2^X$,

$$\Im(A \cup B) = \inf_{x \in (A \cup B)^c} (1 - \delta(\{x\}, A \cup B))$$

$$= \inf_{x \in A^c \cap B^c} ((1 - \delta(\{x\}, A)) \wedge (1 - \delta(\{x\}, B)))$$

$$\geq \inf_{x \in A^c} \inf_{x \in B^c} ((1 - \delta(\{x\}, A)) \wedge (1 - \delta(\{x\}, B)))$$

$$= \inf_{x \in A^c} ((1 - \delta(\{x\}, A)) \wedge \inf_{x \in B^c} (1-\delta(\{x\}, B))$$

$$= \Im(A) \wedge \Im(B)$$

c) For any $\{A_i \backslash i \in I\}$, $\Im(\bigcap_{i \in I} A_i) = \inf_{x \in (\bigcap_i A_i)^c} (1 - \delta(\{x\}, \bigcap_{i \in I} A_i))$

$$\geq \inf_{x \in \bigcup_i A_i^c} (1 - \delta(\{x\}, A_i))$$

$$= \inf_{i \in I} \inf_{x \in A^c} (1 - \delta(\{x\}, A_i))$$

$$= \inf_{i \in I} \Im(A_i)$$

Since $\bigcap_{i \in I} A_i \subset A_i$ implies $\delta(\{x\}, \bigcap_{i \in I} A_i) \leq \delta(\{x\}, A_i) \ \forall i \in I$

Corllarly 3.3.4.

A map $\tau : 2^X \rightarrow I$ which defined by,

$$\tau(A) = \inf_{x \in A} (1 - \delta(\{x\}, A^c))$$

is a fuzzifying topology on X.

Proof.

Clearly byTheorem(2.1) and De'Morgan's Lows

Theorem 3.3.5.

Let (X, δ_1), (Y, δ_2) be two fuzzifying proximity spaces and τ_1, τ_2 be the fuzzifying topology of δ_1, δ_2 respectively. Then for any $f \in Y^X$ we get,

$$\models \tilde{\tilde{C}}(f) \rightarrow C(f)$$

where C is fuzzy continuity w. r. t. τ_1, τ_2.

Proof.

Since $\tau(A) = \inf_{x\in A} (1-\delta(\{x\}, A^c))$

and $\inf_{x\in f^{-1}(A)} (1-\delta_1(f^{-1}(y), f^{-1}(A^c))) \geq \inf_{y\in A} (1-\delta_1(f^{-1}(y), f^{-1}(A^c)))$

$$[C(f)] = \inf_{A\in 2^Y} \min(1, 1-\tau_2(A) + \tau_1(f^{-1}(A)))$$

$$= \inf_{A\in 2^Y} \min(1, 1- \inf_{y\in A}(1-\delta_2(y, A^c))$$

$$+ \inf_{x\in f^{-1}(A)} (1- \delta_1(f^{-1}(y), f^{-1}(A^c))))$$

$$\geq \inf_{A\in 2^Y} \min(1, 1- \inf_{y\in A}(1-\delta_2(y, A^c))$$

$$+ \inf_{y\in A}(1- \delta_1(f^{-1}(y), f^{-1}(A^c))))$$

$$= \inf_{y, A\in 2^Y} \min(1-\delta_2(y, A^c) - \delta_1(f^{-1}(y), f^{-1}(A^c)))$$

$$= [\tilde{\tilde{C}}(f)]$$

3.4. Some propertias on a fuzzifying proximity spaces

Now we explain, how to get a fuzzifying proximity if a fuzzifying uniformity on a set X, is given.

Theorem 3.4.1.

Let (X, U) be a fuzzifying uniform space. Let $\delta_U : 2^X\times 2^X \to I$ be defined by

$$\delta_U(A, B) = \inf_{u[A]\cap B = \phi} (1- U(u)).$$

Then, δ_U is a fuzzifying proximity on X.

Proof.

$fp_1)$ $\delta_U(A, B) = \inf_{u[A]\cap B=\phi} (1-\upsilon(u))$

$= \inf_{u^{-1}[B]\cap A=\phi} (1-\upsilon(u^{-1}))$

$= \delta_U(B, A)$

$fp_2)$ $\delta_U(A\cup B, C) = \inf_{u[A\cup B]\cap C=\phi} (1-\upsilon(u))$

$= \inf_{(u[A]\cup u[B])\cap C=\phi} (1-\upsilon(u))$

$= \inf_{u[A]\cap C=\phi,\ u[B]\cap C=\phi} (1-\upsilon(u))$

$\geq \inf_{u[A]\cap C=\phi} (1-\upsilon(u)) \vee \inf_{u[B]\cap C=\phi} (1-\upsilon(u))$

$= \delta_U(A, C)\vee \delta_U(B, C).$

On the othar hand we prove $\delta_U(A\cup B, C) \leq \delta_U(A, C)\vee \delta_U(B, C)$,

i.e. $\inf_{u[A\cup B]\cap C=\phi} (1-\upsilon(u)) \leq \inf_{u[A]\cap C=\phi} (1-\upsilon(u)) \vee \inf_{u[B]\cap C=\phi} (1-\upsilon(u))$

i.e. $\sup_{u[A\cup B]\cap C=\phi} \upsilon(u) \geq \sup_{u[A]\cap C=\phi} \upsilon(u) \wedge \sup_{u[B]\cap C=\phi} \upsilon(u)$.

We assume that $\sup_{u[A]\cap C=\phi} \upsilon(u) \wedge \sup_{u[B]\cap C=\phi} \upsilon(u) \geq t$, then $\sup_{u[A]\cap C=\phi} \upsilon(u) \geq t$ and $\sup_{u[B]\cap C=\phi} \upsilon(u) \geq t$, hence there exist $u_1, u_2 \in \upsilon$ s.t. $u_1[A]\cap C = \phi$, $u_2[B]\cap C = \phi$, and $\upsilon(u_1) \geq t$, $\upsilon(u_2) \geq t$. Taking $u_3 = u_1 \cap u_2$, since $\upsilon(u_3) \geq \upsilon(u_1) \wedge \upsilon(u_2) \geq t$ and $u_3[A\cup B] \cap C = \phi$, then $\sup_{u[A\cup B]\cap C=\phi} \upsilon(u) \geq t$, therefore $\delta_U(A\cup B, C) \leq \delta_U(A, C)\vee \delta_U(B, C)$.

$fp_3)$ Since $u[\phi] \cap X = \phi$, $\forall u \in \upsilon$ and $\upsilon(X\times X) = 1$, then $\delta_U(\phi, X) = 0$.

$fp_4)$ $\delta_U(A, E)\vee\delta_U(X-E, B) = \inf_{u[A]\cap E=\phi} (1-\upsilon(u)) \vee \inf_{u[X-E]\cap B=\phi} (1-\upsilon(u)) =$

$\inf_{u[A]\cap E=\phi,\ u[X-E]\cap B=\phi} (1-\upsilon(u))$

$\leq \inf_{u[A]\cap B=\phi} (1-\upsilon(u))$

$= \delta_U(A, B).$

$fp_5)$ If $A\cap B = \phi$, then $(A\cap B)(x) \leq \delta_U(A, B)$.

If $A\cap B \neq \phi$, then $u[A]\cap B \neq \phi$,

since $\delta_U(A, B) = \inf_u [(u\in \upsilon \to \sim(u[A]\cap B = \phi)]$

$= \inf_u \min(1, 1-\upsilon(u) + [\sim(u[A]\cap B = \phi)])$

then $(A\cap B)(x) = \delta_U(A, B)$. Therefor $(A\cap B)(x) \leq \delta_U(A, B)$, for any $x\in X$.

Theorem 3.4.2.

Let U be a fuzzifying proximity on X. Then U and δ_U induce the same fuzzifying topology.

Proof.

Let τ_U be a fuzzifying topology iduced by U and $\tau_{\delta U}$ be a fuzzifying topology iduced by δ_U. From Lemma 3.1 in [36] we have

$$\tau_{\delta U} = \inf_{x \in A} (1-\delta_U(\{x\}, A^c))$$

$$= \inf_{x \in A} (1- \inf_{u[x] \subset A} (1- U(u)))$$

$$= \inf_{x \in A} \sup_{u[x] \subset A} U(u) = \tau_U (A).$$

Theorem 3.4.3.

Let U be a fuzzifying uniformity on X, δ_U be a fuzzifying proximity induced by U and $\tau_{\delta U}$ be a fuzzifying topology of δ_U.

Then for any $x \in X$, $A \subset X$,

1) $\models (x\in A^o) \leftrightarrow ((\{x\}, A^c) \notin \delta_U)$.

i.e.

$$A^o(x) = 1- \delta_U (\{x\}, A^c)$$

2) $\models (x\in \bar{A}) = ((\{x\},A) \in \delta_U)$.

i.e.

$$\bar{A} (x) = \delta_U (\{x\}, A)$$

Proof.

1) From Definition 2.1 in [34] and Theorem 3.3 in [36] we have

$$\delta_U(\{x\}, A^c) = \inf_{u[X] \cap A^c = \phi} (1 - U(u))$$

$$= 1 - \sup_{u[x] \cap A^c = \phi} U(u)$$

$$= 1 - \sup_{u[x] = A} U(u)$$

$$= 1 - A^o(x),$$

then $A^o(x) = 1 - \delta_U(\{x\}, A^c)$.

2) From Theorem 2.2. in [34] we have

$$\bar{A}(x) = 1 - (X - A)^o(x)$$

$$= 1 - (1 - \delta_U(\{x\}, A))$$

$$= \delta_U(\{x\}, A).$$

Chapter IV

Fuzzifying Syntopogenous Structures

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Fuzzifying Syntopogenous Structures

Since several papers are common to fuzzy topology, fuzzy proximity and fuzzy uniformity, one would expect, to existence of general structures which include these three concepts. Katsaras gave for the first time in [24] the concept of a fuzzy syntopogenous structure and he showed that, the fuzzy topology, the fuzzy proximity and the fuzzy uniformity are special cases of fuzzy syntopogenous structures. Also in [26] he introduced a new definition of a fuzzy syntopogenous structure in the embedding of the structure in a multiple valued logic, and he showed that there is a one- to- one correspondence between the Artico- Moresco fuzzy proximities [4] and the symmetrical fuzzy topogenous structures. Mingsheng Ying introduced three papers [33-35] about the concept of fuzzifying topology. Also in [36] he introduced the concept of fuzzifying uniform spaces. We introduced in chapter three the concept of fuzzifying proximity spaces. In this chapter we introduce the concept of fuzzifying syntopogenous structures and some fundamental properties are established. We give a natural links between fuzzifying syntopogenous structures, fuzzifying topology, fuzzifying proximity and fuzzifying uniformity.

4. 1. Fuzzifying syntopogenous structures

Definition 4.1.1.

A fuzzifying topogenous order on a set X is a function

$$\eta: 2^X \times 2^X \to I,$$

which satisfies:

1) $\eta(\phi, \phi) = \eta(X, X) = 1$,

2) $\eta(A, B) \leq (1 - A(x)) \vee B(x)$ for all $x \in X$,

3) $A \subset C$, $D \subset B$ implies $\eta(C, D) \leq \eta(A, B)$,

4) $\eta(A \cup B, C) = \eta(A, C) \wedge \eta(B, C)$,

$\eta(A, B \cap C) = \eta(A, B) \wedge \eta(A, C)$

Definition 4.1.2.

If η_1 and η_2 are topogenous orders (fuzzifying topogenous orders) on X such that $\eta_1 \subset \eta_2 (\forall A, B \in 2^X \times 2^X, \eta_1(A, B) \leq \eta_2(A, B))$, then we say that η_1 is finer than η_2 or η_2 is coarser than η_1.

We omit the proof of the following easily established Lemmas.

Lemma 4.1.3.

Let η be a fuzzifying topogenous order on X and let η^c be defined by $\eta^c(A,B) = \eta(B^c, A^c)$. Then, η^c is a fuzzifying topogenous order on X.

Lemma 4.1.4.

Let η be a fuzzifying topogenous order on X, $A \subset X$ and $x \in X$. Then:

1) $\eta(X, \phi) = 0$; $\eta(\phi, A)$ and $\eta(A, X) = 1$,

2) $\eta(x, X) = 1$; $\eta(x, \phi) = 0$ and $\eta(\phi, X) = 1$,

3) $\eta(X, x^c) = 0$; $\eta(\phi, x^c) = 1$.

Definition 4.1.5.

A fuzzifying topogenous order η on X is called:

1) Perfect if $\eta(\bigcup_i A_i, B) = \inf_i \eta(A_i, B)$

2) Biperfect if it is perfect and $\eta(A, \bigcap_i B_i) = \inf_i \eta(A, B_i)$

3) Symmetrical if $\eta = \eta^c$

Lemma 4.1.6.

η is biperfect iff both η and η^c are perfect.

Proof.

Cleary, from Lemma (4.1.4). Definition (4.1.5) and De' Morgan's Lows.

Definition 4.1.7.

If η_1, η_2 are fuzzifying topogenous orderson X,

then $\eta = \eta_1 o\, \eta_2$ defined by

$$\eta(A, B) = \sup_{C \subset X} (\eta_2(A, C) \wedge \eta_1(C, B))$$

Lemma 4.1.8.

let η_1, η_2 be fuzzifying topogenous orders on X and let

$\eta = \eta_1 o \eta_2$. Then:

I) η is a fuzzifying topogenous order on X.

II) If η_o is a fuzzifying topogenous order with $\eta_o \geq \eta_1, \eta_2$ then $\eta_o \geq \eta$

III) $\eta^c = \eta_2^c \circ \eta_1^c$

Proof.

I) 1) Clearly.

2) To prove $\eta(A, B) \leq (1- A(x)) \vee B(x)$ for all $x \in X$ we prove only $\eta(A, B) = 0$, if $x \in A$, $x \notin B$. Since either $x \in C$ or $x \notin C$ then we have two cases, first $x \in C$, $x \notin B$. But $\eta_1(C, B) \leq (1-C(x)) \vee B(x)$, so $\eta_1(C, B) = 0$, then $\eta(A, B) = 0$. On the other hand $x \notin C$, $x \in A$. Since, $\eta_2(A, C) \leq (1- A(x)) \vee C(x)$ hence $\eta_2(A, C) = 0$, then $\eta(A, B) = 0$.

3) Let $A_1 \subset A$, $B \subset B_1$. Since $\eta_2(A, C) \leq \eta_2(A_1, C)$;

$\eta_1(C, B) \leq \eta_1(C, B_1)$ thus,

$$\eta_2(A, C) \wedge \eta_1(C, B) \leq \eta_2(A_1, C) \wedge \eta_1(C, B_1), \text{ and}$$

$$\sup_{C \subset X} \eta_2(A, C) \wedge \eta_1(C, B) \leq \sup_{C \subset X} (\eta_2(A_1, C) \wedge \eta_1(C, B_1)).$$

Then $\eta(A, B) \leq \eta(A_1, B_1)$.

4) $$\eta(A \cup B, C) = \sup_{D \subset X} (\eta_2(A \cup B, D) \wedge \eta_1(D, C))$$

$$\leq \sup_{D \subset X} \left((\eta_2(A, D) \wedge \eta_2(B, D)) \wedge \eta_1(D, C)\right)$$

$$= \sup_{D \subset X} \left(\eta_2(A, D) \wedge \eta_1(D, C)\right)$$

$$\wedge \sup_{D \subset X} \left(\eta_2(B, D) \wedge \eta_1(D, C)\right)$$

$$= \eta(A, C) \wedge \eta(B, C).$$

Similarly $\eta(A, B \cap C) = \eta(A, B) \wedge \eta(A, C)$

II) Let $A, B \in 2^X$. We need to show that

$\eta_o(A, B) \geq \eta(A, B)$. If $\eta(A, B) = 0$, there is nothing to prove. Suppose that $\eta(A, B) \geq 0$ and let $t \in (0,1]$, $\eta(A, B) \geq t$. There exists $C \subset X$ such that , $\eta_2(A, C) \wedge \eta_1(C, B) \geq t$, implies $\eta_2(A, C) \geq t$ and $\eta_1(C, B) \geq t$. But $\eta_2(A, C) \leq (1-A(x)) \vee (C(x)) = 1-(A(x) \wedge (1-C(x))) = 1-(A \cap C^c)(x)$ for all $x \in X$. Then $1-(A \cap C^c)(x) \geq t$ for all $x \in X$, $t \in (0,1]$. Take $t = 1$, then $(A \cap C^c)(x) = 0$, For all $x \in X$, then $A \subset C$, Since $\eta_0(A, B) \geq \eta_0(C, B)$.

$\eta_0 \geq \eta_1$, then $\eta_o(A, B) \geq t$. Now, we have $\eta(A, B) \geq t$ implies $\eta_o(A, B) \geq t$. Then $\eta_o(A, B) \geq \eta(A, B)$.

III) $\eta^c(A, B) = \eta(B^c, A^c)$

$$= \sup_{C \subset X}\left(\eta_2(B^c, C) \wedge \eta_1(C, A^c)\right)$$

$$= \sup_{C \subset X}\left(\eta_2^c(C^c, B) \wedge \eta_1^c(A, C^c)\right)$$

$$= \sup_{D \subset X}\left(\eta_2^c(D, B) \wedge \eta_1^c(A, D)\right)$$

$$= \eta_2^c \circ \eta_1^c(A, B)$$

and so $\eta^c = \eta_2^c \circ \eta_1^c$

Definition 4.1.9.

A fuzzifying syntopogenous structure on a non- empty set X is a non- empty family S of fuzzifying topogenous orders on X satisfying the following axioms.

FS_1) S is directed in the sense that given $\eta_1, \eta_2 \in S$ there exists $\eta \in S$ with $\eta \geq \eta_1, \eta_2$

FS_2) Given $\eta \in S$ and $\in > 0$ there exists $\eta^* \in S$ with $\eta^* \circ \eta^* + \in \geq \eta$.

Difinition 4.1.10.

If a fuzzifying syntopogenous structure S on X consists of a single fuzzifying topogenous order, then S is called a fuzzifying topogenous, (X, S) a fuzzifying topogenous space.

Lemma 4.1.11.

If $S = \{\eta\}$ is a fuzzifying topogenous structure on X, then $\eta = \eta \circ \eta$.

Proof.

Cleary from Definition (4.1.7) and (by (II) of Lemma (4.1.9)).

Definition 4.1.12.

A fuzzifying syntopogenous structure S is called perfect (resp. Biperfect, resp. Symmetrical), if every member of S is perfect (resp. Biperfect, resp. Symmetrical).

Lemma. 4.1.13.

Let S be a fuzzifying syntopogenous structure on X and define η_S by

$$\eta_S(A, B) = \sup\{\eta(A, B) \mid \eta \in S\}.$$

Then, $S' = \{\eta_S\}$ is a fuzzifying topogenous structure on X.

Proof.

1) Clearly $\eta_S(\phi, \phi) = \eta_S(X, X) = 1$

2) To prove $\eta_S(A, B) \leq (1-A(x)) \vee B(x)$ for all $x \in X$, we prove only $\eta_S(A, B) = 0$, if $x \in A$, $x \notin B$. Since $\eta(A, B) \leq (1-A(x)) \vee B(x)$, for all $x \in X$, hence $\eta(A, B) = 0$ in this case for all $\eta \in S$. Then $\eta_S(A, B) = 0$, and so $\eta_S(A, B) \leq (1-A(x)) \vee B(x)$ for all $x \in X$.

3) If $A \subset C$, $D \subset B$, then $\eta(C, D) \leq \eta(A, B)$ for all $\eta \in S$. Since $\sup\{\eta(C, D), \eta \in S\} \leq \sup\{\eta(A, B), \eta \in S\}$, then $\eta_S(C, D) \leq \eta_S(A, B)$.

4) $\eta_S(A \cup B, C) = \sup\{\eta(A \cup B, C) \setminus \eta \in S\}$

$$= \sup\{\eta(A, C) \wedge \eta(B, C) \setminus \eta \in S\}$$

$$= (\sup\{\eta(A, C) \setminus \eta \in S\}) \wedge (\sup\{\eta(B, C) \setminus \eta \in S\})$$

$$= \eta_S(A, C) \wedge \eta_S(B, C).$$

4.2. The cuts and the representation of a fuzzifying topogenous order.

Now we study the cuts of a fuzzifying topogenous order η let us denote.

$$\eta_\alpha = \{ (A, B) \in 2^X \times 2^X \setminus \eta (A, B) \geq \alpha\}$$

η_α said to be weak α-cut of a fuzzifying topogenous order

Lemma 4.2.1.

Let η be a fuzzifying topogenous order on X, then for any α in I, η_α is a topogenous order on X .Moreover, if $\alpha_1 \geq \alpha_2$ then $\eta_{\alpha_1} \subset \eta_{\alpha_2}$.

Proof.

1) Since $\eta(\phi, \phi) = \eta(X, X) = 1 \geq \alpha$, then (ϕ, ϕ); $(X, X) \in \eta_\alpha$

2) $(A, B) \in \eta_\alpha \Rightarrow \eta(A, B) \geq \alpha$ for all $\alpha \in I$

$\Rightarrow (1 - A(x)) \vee B(x) \geq \alpha$ for all $\alpha \in I$, for all $x \in X$

$\Rightarrow 1 - (A(x) \wedge (1-B(x))) \geq \alpha$, for all $\alpha \in I$, for all $x \in X$

take $\alpha = 1$, then $[\sim((x\in A) \wedge (x \notin B))] = 1$ for all $x \in X$

$\Rightarrow [x\in A \rightarrow x \in B] = 1$, for all $x \in X$

$\Rightarrow A \subset B$.

3) Let $A \subset C$, $D \subset B$ and $(A, B) \in \eta_\alpha$ implies, $\eta(A, B) \geq \alpha$, hence,

$\alpha \leq \eta (A, B) \leq \eta(A_1, B_1)$, then $(A_1, B_1) \in \eta_\alpha$.

4) $(A\cup B, C) \in \eta_\alpha \Leftrightarrow \eta(A\cup B, C) \geq \alpha$

$\Leftrightarrow \eta (A, C) \wedge \eta(B, C) \geq \alpha$

$\Leftrightarrow \eta (A, C) \geq \alpha, \eta (B, C) \geq \alpha$

$\Leftrightarrow (A, C) \in \eta_\alpha, (A, C) \in \eta_\alpha$.

Similary $(A, B\cap C) \in \eta_\alpha \Leftrightarrow (A, B) \in \eta_\alpha, (A, C) \in \eta_\alpha$.

Hence η_α is a topogenous order on X.

On the other hand, take $(A, B)\in \eta_{\alpha_1}$, so $\eta(A, B) \geq \alpha_1$, but $\alpha_1 \geq \alpha_2$, hence $\eta(A, B) \geq \alpha_2$, then $(A, B) \in \eta_2$.

On the contrary, for the representation theorem, we would like to see that if the axioms for a fuzzifying topogenuous order are effectively satisfied when we start from a family of topogenous orders associated with every α in I .More precisely.

Lemma 4.2.2.

Let $\{\eta_\alpha \backslash\ \alpha \in I\}$ be a family of topogenous orders on X s.t. $\alpha_1 \geq \alpha_2$ implies $\eta_{\alpha_1} \subset \eta_{\alpha_2}$. Let η be a fuzzy relation built by

$$\eta(A, B) = \sup\{\alpha\backslash (A, B)\in \eta_\alpha\}.$$

Then η is a fuzzifying topogenous order on X.

Proof.

1) Since $(\phi, \phi);(X, X) \in \eta_1$, then $\eta(\phi, \phi) = \eta(X, X) = 1$

2) $\eta(A, B) = \sup \{\alpha|\ (A, B)\in \eta_\alpha\}$

$\leq \sup \{\alpha|\ A\subset B\}$

$= [A\subset B]$

$= (1- A(x)) \vee B(x)$, for all $x \in X$.

3) For $A\subset C$, $D\subset B$, $\eta(C, D) = \sup\{\alpha|\ (C, D)\in \eta_\alpha\}$

$\leq \sup\{\alpha|\ (A, B)\in \eta_\alpha\}$

$= \eta(A, B)$.

4) $\eta(A\cup B, C) = \sup\{\alpha|\ (A\cup B, C)\in \eta_\alpha\}$

$= \sup\{\alpha|\ (A, C)\in \eta_\alpha,(B, C)\in \eta_\alpha\}$

$= \sup\{\alpha|\ (A, C)\in \eta_\alpha\}\wedge \sup\{\alpha|\ (B, C)\in \eta_\alpha\}$

$= \eta(A, C) \wedge \eta(B, C)$.

Similary $\eta(A, B \cap C) = \eta(A, B) \wedge \eta(A, C)$.

Theorem 4.2.3.

Let η be a fuzzifying topogenous order on X and η_α be the weak α-cut as defined precedingly. From the families of topogenous η_α on X we build $\eta^*$by

$$\eta^*(A, B) = \sup\{\alpha | (A, B) \in \eta_\alpha\}.$$

Then $\eta = \eta^*$

Proof.

The proof is trivial from the preceding results and the well known fact that .

$$\sup\{\alpha | (A, B) \in \eta_\alpha\} = \sup\{\alpha | \eta(A, B) \geq \alpha\} = \eta(A, B).$$

4.3. Some properties of fuzzifying syntopogenous structures.

In this section, we explain the natural links between fuzzifying syntopogenous structures, fuzzifying topology, fuzzifying proximity and fuzzifying uniformity.

Theorem 4.3.1.

Let η be a fuzzifying topogenous order on X and $\tau_\eta : 2^X \to I$ defined by:

$$(A \in \tau_\eta) := (\forall x)(x \in A \to (x, A) \in \eta)$$

i.e.

$$\tau_\eta(A) = \inf_{x \in A} \eta(x, A).$$

Then, τ_η is a fuzzifying topolgy on X.

Proof.

1) Since $\eta(X, X) = 1$, then $\eta(x, X) \geq \eta(X, X)$ thus $\eta(x, X) = 1$. Then $\tau_\eta(X) = 1$.

2) $$\tau_\eta(A \cap B) = \inf_{x \in A \cap B} \eta(x, A \cap B)$$

$$= \inf_{x \in A \cap B} (\eta(x, A) \wedge \eta(x, B))$$

$$\geq \inf_{x \in A} \eta(x, A) \wedge \inf_{x \in B} \eta(x, B)$$

$$= \tau_\eta(A) \wedge \tau_\eta(B).$$

3) Since $\eta(x, \bigcup_{i \in I} A_i) \geq \eta(x, A_i)$, then,

$$\tau_\eta(\bigcup_{i \in I} A_i) = \inf_{x \in \bigcup_{i \in I} A_i} \eta(x, \bigcup_{i \in I} A_i)$$

$$\geq \inf_{i \in I} \inf_{x \in A_i} \eta(x, A_i)$$

$$= \inf_{i \in I} \tau_\eta(A_i).$$

Now, let δ be a fuzzifying proximity on X. Define η by $\eta(A, B) = 1 - \delta(A, B^c)$. It is easy to see that η is a symmetrical fuzzifying topogenous order on X and that $S_\delta = \{\eta\}$ is a fuzzifying topogenous structure with $\tau_{S_\delta} = \tau_\delta$. If $S = \{\eta\}$ is a symmetrical fuzzifying topogenous structure on X, then the function.

$$\delta: 2^X \times 2^X \to I, \quad \delta(A, B) = 1 - \eta(A, B^c),$$

is fuzzifying proximity on X with $S = S_\delta$. Thus we have:

Theorem 4.3.2

The mapping $\delta \to S_\delta$, from the set of all fuzzifying proximities on X to the set of all symmetrical fuzzifying topogenous structures on X, is one-to-one and onto. Moreover $\tau_{S_\delta} = \tau_\delta$.

Theorem 4.3.3

Let U be a fuzzifying uniformity on X. Difine η_U by.

$$\eta_U(A,\ B) = \sup_{u[A]\cap B^c = \phi} U(u)$$

Then η_U is a biperfect fuzzifying topogenous order on X. Moreover,

$\tau_\eta = \tau_U = \tau_\delta$.

Proof.

1) Take $u = X \times X$, since $\phi = u[\phi]$ and $U(X \times X) = 1$, then $\eta(\phi, \phi) = 1$

similarly $u = X \times X$, $X = u[X]$; $U(X \times X) = 1$ implies $\eta(X \times X) = 1$

2) To prove $\eta(A, B) \le (1 - A(x)) \vee B(x)$ for all $x \in X$ we prove only $\eta(A, B) = 0$, if $x \in A$, $x \notin B$. Since $x \in A$ implies $x \in u[A]$. But $x \in B^c$, then $u[A] \cap B^c \ne \phi$ for all $u \in U$ and so $\eta(A, B) = 0$.

3) If $A \subset C$, $D \subset B$, Since $u[A] \subset u[C]$, $B^c \subset D^c$, then $u[A]\cap B^c \subset u[C] \cap D^c$ then,

$$\eta(C, D) = \sup_{u[C]\cap D^c = \phi} U(u)$$

$$\le \sup_{u[A]\cap B^c = \phi} U(u)$$

$$= \eta(A,\ B).$$

4) $$\eta(A \cup B,\ C) = \sup_{u[A\cup B]\cap C^c = \phi} U(u)$$

$$= \sup_{u[A]\cap B^c=\phi,\ u[B]\cap C^c=\phi} U(u)$$

$$= \sup_{u[A]\cap B^c=\phi} U(u) \wedge \sup_{u[B]\cap C^c=\phi} U(u)$$

$$= \eta(A,\ B) \wedge \eta(B,\ C).$$

Similary, $\eta(A,\ B\cap C) = \eta(A,\ B) \wedge \eta(A,\ C)$.

Now we proof $\tau_\eta = \tau_U = \tau_\delta$.

$$\tau_\eta(A) = \inf_{x\in A} \eta(x, A)$$

$$= \inf_{x\in A} \sup_{u[x]\cap A^c=\phi} U(u)$$

$$= \inf_{x\in A} \sup_{u[x]\subset A} U(u)$$

$$= \tau_U(A).$$

And from Theorem 4.3.2 we have $\tau_\eta = \tau_\delta$.

Finally, let $A = \bigcup_{j\in J} A_j$, then $u[A] = \bigcup_{j\in J} u[A_j]$ and so

$$\eta_U(A, B) = \sup_{\bigcup_{j\in J} u[A_j]\cap B^c=\phi} U(u)$$

$$= \inf_{j\in J} \left(\sup_{u[A_j]\cap B^c=\phi} U(u)\right)$$

$$= \bigwedge_{j\in J} \eta_U(A_j, B).$$

In an analogous way we show that

$$\eta_U(A, \bigcap_{j\in J} B) = \bigwedge_{j\in J} \eta_U(A, B_j),$$

and so η_U is a biperfect fuzzifying topogenous order on X.

Theorem 4.3.4

Let N_x be fuzzifying neighborhood system of $x \in X$. Then

$$\eta(A, B) = \inf_{x \in X} ((1-A(x)) \vee N_x(B))$$

is a fuzzifying topogenous order on X.

Moreover, $N_x(A) = \eta(x, A)$ and $\tau_N = \tau_\eta$.

Proof.

1) $\eta(\phi, \phi) = \inf_{x \in X} X(x) \vee N_x(A) = 1.$

$\eta(X, X) = \inf_{x \in X} (1- X(x)) \vee N_x(X) = 1.$

2) $\eta(A, B) = \inf_{x \in X} ((1-A(x)) \vee N_x(B))$

$\leq (1-A(x)) \vee B(x)$

3) $A \subset C$, $D \subset B$ implies

$$\eta(C, D) = \inf_{x \in X} ((1-C(x)) \vee N_x(D))$$

$$\leq \inf_{x \in X} ((1-A(x)) \vee N_x(B))$$

$$= \eta(A, B).$$

4) $\eta(A \cup B, C) = \inf_{x \in X} ((1- (A \cup B)(x)) \vee N_x(C)).$

$$= \inf_{x \in X} (((1- A(x)) \wedge (1-B(x))) \vee N_x(C)).$$

$$= \inf_{x \in X} ((1- A(x)) \vee N_x(C)) \wedge \inf_{x \in X} ((1- B(x)) \vee N_x(C))$$

$$= \eta(A, C) \wedge \eta(B, C).$$

Now, $\eta(x, A) = \inf_{x \in X} ((1-x(x)) \vee N_x(A)) = N_x(A).$

Finally, $\tau_\eta(A) = \inf_{x \in A} \eta(x, A)$

$= \inf_{x \in A} N_x(A)$

$= \tau_N(A)$

Corallary 4.3.5.

The fuzzifying topogenous order η can be constructed from the cuts U_α, $\alpha > 0$, of the fuzzifying uniformity by use of the equality,

$$\eta(A, B) = \sup_{\alpha > 0} (\eta_\alpha(A, B) \wedge \alpha),$$

where $\eta_\alpha(A, B)$ is defined from U_α by:

$(A, B) \in \eta_\alpha := (\exists u)(u \in U_\alpha \wedge u[A] \cap B^c = \phi).$

Proof.

$$\eta(A, B) = \sup_{u} \{U(u) : u[A] \cap B^c = \phi\}$$

$$= \sup_{\alpha > 0} \{\sup \{U(u) : u[A] \cap B^c = \phi\} \wedge \alpha : U(u) \geq \alpha\}$$

$$= \sup_{\alpha > 0} \{\sup \{u \in U_\alpha : u[A] \cap B^c = \phi\} \wedge \alpha\}$$

$$= \sup_{\alpha > 0} \{\eta_\alpha(A, B) \wedge \alpha\}.$$

Corllary 4.3.6

To every α we can associate with U_α a topogenous order η_α by taking,

$$\eta_\alpha(A, B) := (\exists u)(u \in U \wedge u[A] \cap B^c = \phi), \text{ and}$$

$$\eta(A, B) = \sup \{\alpha > 0 : \eta_\alpha(A, B)\}.$$

4.4. Fuzzifying topogenous continuity

Definition 4.4.1.

Let η_1 , η_2 be two fuzzifying tepogenous orders. A unary fuzzy predicate $c^* \in \Im(Y^X)$, called fuzzy topogenous continuity is defined as,

$$C^*(f) := (\forall C,\ D)\Big((C,\ D) \in \eta_2 \rightarrow \left(f^{-1}(C),\ f^{-1}(D)\right) \in \eta_1\Big)$$

Intuitively the degree to which f is continuous is,

$$\left[C^*(f)\right] = \inf_{C, D, \in 2^Y} \min\left(1,\ 1 - \eta_2(C,\ D) + \eta_1(f^{-1}(C),\ f^{-1}(D))\right)$$

Lemma 4.4.2.

Let δ_1, δ_2 be two fuzzifying proximitiy on X, Y respectively, then for any $f \in Y^X$ we get

$$\models C^*(f) \leftrightarrow \overset{\approx}{C}(f)$$

Where $\overset{\approx}{C}$ is fuzzy continuity w.r.t. η_{δ_1}, η_{δ_2} respectively.

Proof.

Since $\eta(A, B) = 1 - \delta(A, B^c)$, then

$$\left[C^*(f)\right] = \inf_{C, D \in 2^Y} \min\Big(1,\ 1 - \eta_2(C,\ D) + \eta_1\left(f^{-1}(C),\ f^{-1}(D)\right)\Big)$$

$$= \inf_{C, D \in 2^Y} \min\Big(1,\ 1 - \delta_1\left(f^{-1}(C),\ f^{-1}(D)\right) + \delta_2\left((C,\ D^c)\right)\Big)$$

$$= \left[\overset{\approx}{C}(f)\right].$$

Lemma 4.4.3.

Let η_1, η_2 and η_3 be three fuzzifying topogenous orders on X, Y and Z respectively. For any $f \in Y^X$ and $g \in Y^Z$

$$1) \quad \models C^*(f) \to \left(C^*(g) \to C^*(gof)\right)$$

$$2) \quad \models C^*(g) \to \left(C^*(f) \to C^*(fog)\right).$$

Proof.

We only demonstrate (1). It suffices to show that,

$$\left[C^*(f)\right] \le \left[C^*(g) \to C^*(gof)\right].$$

If $\left[C^*(g)\right] \le \left[C^*(gof)\right]$, its obvious.

If $\left[C^*(g)\right] > \left[C^*(gof)\right]$, then

$$\left[C^*(g)\right] - \left[C^*(gof)\right]$$

$$= \inf_{C,D \in 2^Z} \min\left(1, 1 + \eta_2\left(g^{-1}(C), g^{-1}(D)\right) - \eta_3(C, D)\right)$$

$$- \inf_{C,D \in 2^Z} \min\left(1,\ 1 + \eta_1\left((gof)^{-1}(C),\ (gof)^{-1}(D)\right) - \eta_3(C,\ D)\right)$$

$$\le \sup_{C,\ D \in 2^Z} \left(\eta_2\left(g^{-1}(C),\ g^{-1}(D)\right) - \eta_1\left((gof)^{-1}(C),\ (gof)^{-1}(D)\right)\right)$$

$$\le \sup_{A,\ B \in 2^Y} \left(\eta_2(A,\ B) - \eta_1\left(f^{-1}(A),\ f^{-1}(B)\right)\right)$$

therefore,

$$\left[C^*(g) - C^*(gof)\right] = \min\left(1,\ 1 - \left[C^*(g)\right] + \left[C^*(gof)\right]\right)$$

$$\geq \inf_{A,B\in 2^Y} \min\left(1,\ 1-\eta_2(A,\ B)+\eta_1\left(f^{-1}(A),\ f^{-1}(B)\right)\right)$$
$$=\left[C^*(f)\right]$$

Hence $\left[C^*(g)\to C^*(gof)\right]\geq\left[C^*(f)\right]$.

Lemma 4.4.4.

Let η_1, η_2 be two fuzzifying topogenous orders on X, Y respectively and τ_{η_1}, τ_{η_2} be the fuzzifying topologies of η_1, η_2 respectively. Then, for any $f\in Y^X$ we get

$$\models C^*(f)\to C(f)$$

where C is fuzzy continuity w.r.t. τ_{η_1}, τ_{η_2}.

Proof.

We prove that $\left[C^*(f)\right]\leq\left[C(f)\right]$.

Since $\tau_\eta(A)=\inf_{x\in A}\eta(x,\ A)$

and $\inf_{x\in f^{-1}(A)}\eta_1\left(x,\ f^{-1}(A)\right)\geq\inf_{y\in A}\eta_1\left(x,\ f^{-1}(A)\right)$, then

$$\left[C(f)\right]=\inf_{A\in 2^Y}\min\left(1,\ 1-\tau_{\eta_2}(A)+\tau_{\eta_1}\left(f^{-1}(A)\right)\right)$$
$$=\inf_{A\in 2^Y}\min\left(1,\ 1-\inf_{y\in A}\eta_2(y,\ A)+\inf_{x\in f^{-1}(A)}\eta_1(x,\ f^{-1}(A)\right)$$
$$\geq\inf_{A\in 2^Y}\min\left(1,\ 1-\inf_{y\in A}\eta_2(y,\ A)+\inf_{y\in A}\eta_1(x,\ f^{-1}(A)\right)$$
$$\geq\inf_{y\in A\in 2^Y}\min\left(1,\ 1-\eta_2(y,\ A)+\eta_1(f^{-1}(y,\ f^{-1}(A)\right)$$
$$=\left[C^*(f)\right]$$

4.5. Inverse image of a fuzzifying topogenous order.

Definition 4.5.1.

Let $f: X \to Y$ be a function and let η be a fuzzifying topogenous order on Y. The mapping $\eta_1 : 2^X \times 2^X \to I$, $\eta_1(A, B) = \eta(f(A), (f(B^c))^c)$ is called the inverse image of η by the mapping f and is denoted by $f^{-1}(\eta)$

Theorem 4.5.2.

$f^{-1}(\eta)$ is a fuzzifying topogenous order on X, where $f: X \xrightarrow{\text{onto}} Y$ and η is a fuzzifying topogenous order on Y.

Proof.

1) $f^{-1}(\eta)(\phi, \phi) = \eta(f(\phi), (f(X))^c) = \eta(\phi, \phi) = 1$,

$f^{-1}(\eta)(X, X) = \eta(f(X), (f(\phi))^c) = \eta(Y, Y) = 1$.

2) To prove $f^{-1}(\eta)(A, B) \leq (1 - A(x)) \vee B(x)$ for all $x \in X$, we prove only $\eta_1(A, B) = 0$ if $x \in A$, $x \notin B$. Since $f(A) \cap (f(B^c))^c = \phi$, then $f^{-1}(\eta)(A, B) = 0$.

3) For $A \subset C$, $D \subset B$, $f^{-1}(\eta)(A, B) = \eta(f(A), (f(B^c))^c)$

$$\geq \eta f(C), (f(D^c))^c)$$
$$= f^{-1}(\eta)(C, D)$$

4) $f^{-1}(\eta)(A \cup B, C) = \eta(f(A \cup B), (f(C^c))^c)$

$$= \eta(f(A) \cup f(B), (f(C^c))^c)$$
$$= \eta(f(A), (f(C^c))^c) \wedge \eta(f(B), (f(C^c))^c)$$
$$= f^{-1}(\eta)(A, C) \wedge f^{-1}(\eta)(B, C),$$

$$f^{-1}(\eta)(A,\ B\cap C)=\eta(f(A),\ (f(B^c\cup C^c))^c)$$

$$=\eta(f(A),\ (f(B^c)\cup f(C^c))^c)$$

$$=\eta(f(A),\ (f(B^c))^c\wedge(f(C^c))^c)$$

$$=\eta(f(A),\ (f(B^c))^c)\wedge(f(A),\ (f(C^c))^c)$$

$$=f^{-1}(\eta)(A,\ B)\wedge f^{-1}(\eta)(A,\ C).$$

Lemma 4.5.3.

Let f be a surjective function from X to Y, and η_1, η_2 be two fuzzifying topogenous orders on Y. Then

1) For A, B $\in 2^Y$ we have $f^{-1}(\eta_1)(f^{-1}(A),\ f^{-1}(B))=\eta_1(A,\ B)$.

2) If $\eta_1\subset\eta_2$, then $f^{-1}(\eta_1)\subset f^{-1}(\eta_2)$.

3) If $\{\eta_i : i\in I\}$ is a family of fuzzifying topogenous orders on Y,

then $f^{-1}(\bigcup_i \eta_i)=\sup_i(f^{-1}(\eta_i))$..

4) If η is perfect (resp. biperfect), then $f^{-1}(\eta)$ is perfect (resp. biperfect).

5) $(f^{-1}(\eta))^c=f^{-1}(\eta^c)$.

6) If η is symmetrical, then $f^{-1}(\eta)$ is also symmetrical.

7) $f^{-1}(\eta_1 o \eta_2)=f^{-1}(\eta_1) o f^{-1}(\eta_2)$.

Proof.

1) $f^{-1}(\eta_1)(f^{-1}(A),\ f^{-1}(B))=\eta_1(f(f^{-1}(A)),\ (f((f^{-1}(B))^c))^c)$

$$=\eta_1(A,\ (f(f^{-1}(B^c)))^c)$$

$$=\eta_1(A,\ B).$$

2) $f^{-1}(\eta_1)(A, B) = \eta_1(f(A), (f(B^c))^c)$

$$\leq \eta_2(f(A), (f(B^c))^c)$$

$$= f^{-1}(\eta_2)(A, B).$$

3) $f^{-1}(\bigcup_i \eta_i)(A, B) = (\bigcup_i \eta_i)(f(A), (f(B^c))^c)$

$$= \sup_i \eta_i(f(A), (f(B^c))^c) = \sup_i f^{-1}(\eta_i).$$

4) $f^{-1}(\eta)(\bigcup_i A_i, B) = \eta(f(\bigcup_i A_i), (f(B^c))^c)$

$$= \eta(\bigcup_i f^{-1}(A_i), (f(B^c))^c)$$

$$= \inf_i \eta(f(A_i), (f(B^c))^c)$$

$$= \inf_i f^{-1}(\eta)(A_i, B).$$

Similarly $f^{-1}(\eta)(A, \bigcap_i B_i) = \inf_i f^{-1}(\eta)(A, B_i)$

5) $(f^{-1}(\eta))^c(A, B) = f^{-1}(\eta)(B^c, A^c)$

$$= \eta(f(B^c), (f(A))^c)$$

$$= \eta^c(f(A), (f(B^c))^c)$$

$$= f^{-1}(\eta^c)(A, B).$$

6) Since, $(f^{-1}(\eta))^c = f^{-1}(\eta^c)$, $\eta = \eta^c$.

Then $f^{-1}(\eta)$ is symmetrical.

7) Let A, $B \in 2^X$, $C \in 2^Y$, and $D = f^{-1}(C)$. Then

$f^{-1}(\eta_2)(A, D) = \eta_2(f(A), (f(D^c))^c) = \eta_2(f(A), C)$

$f^{-1}(\eta_1)(D, B) = \eta_1(f(D), (f(B^c))^c) = \eta_1(C, (f(B^c))^c)$.

Hence $f^{-1}(\eta_1 o \eta_2)(A, B) = \eta_1 o \eta_2(f(A), (f(B^c))^c)$

$$= \sup_{C \subset Y}(\eta_2\big(f(A), C\big) \wedge \eta_1(C, (f(B^c))^c))$$

$$= \sup_{D \subset X}(f^{-1}(\eta_2)(A, D) \wedge f^{-1}(\eta_1)(D, B))$$

$$= f^{-1}(\eta_1) o f^{-1}(\eta_2)$$

and so $f^{-1}(\eta_1 o \eta_2) = f^{-1}(\eta_1) o f^{-1}(\eta_2)$.

Lemma 4.5.4.

Let $f: X \to Y$ and $g: Y \to Z$ be functions and let η be a fuzzifying topogenous order on Z. Then

$$(gof)^{-1}(\eta) = f^{-1}\left(g^{-1}(\eta)\right).$$

Proof.

For $A, B \in 2^X$, we have

$$\begin{aligned}(gof)^{-1}(\eta)(A,\ B) &= \eta((gof)(A),\ ((gof)(B^c))^c)\\ &= \eta(g(f(A)),\ (g(f(B^c))^c))\\ &= \eta(g(f(A)),\ (g(f(B^c))^{cc})^c)\\ &= g^{-1}(\eta)(f(A),\ (f(B^c))^c)\\ &= f^{-1}(g^{-1}(\eta))(A,\ B).\end{aligned}$$

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Arabic Summary

عن الأبنية التوبولوجية الزُغبية

رسالة مقدمة إلى

كلية العلوم بالفيوم - جامعة القاهرة

للحصول على

درجة ماجستير العلوم فى الرياضيات (توبولوجى)

مقدمة من

مصطفى الدمرديرى أحمد حسين

بكالوريوس علوم (رياضيات) ١٩٩٤

إشراف

الدكتور

محمود صبرى محمد سيف

كلية العلوم بالفيوم - جامعة القاهرة

الدكتور

سلامة نادى محمود الديب

كلية العلوم بالفيوم - جامعة القاهرة

الدكتور

محمد خليل حسن الجبار

كلية الهندسة ببورسعيد - جامعة قناة السويس

١٩٩٧

بسم الله الرحمن الرحيم

الملخص العربى

معظم الصفات التى يهتم بها العلم الحديث والتى تقابلنا فى حياتنا العملية تفتقر إلى التحديد التام أمثال تلك الصفات جيد وردىء وطويل جدا وغزير . وعدم التحديد الذى يهمنا هنا ليس من النوع الإحصائى بل هو عدم تحديد متأصل فى طبيعة تلك الصفات . ولغياب التحديد عن مثل تلك الصفات استحال فى السابق الحصول لكل واحدة منها على مجموعة الأشياء التى تحقق تلك الصفة .

وكان المنطق عديد القيم *(Multiple-valued logic)* قد بدأ منذ أكثر من خمسين عاما عند التعامل مع الصفات الناقصة التحديد ، بأن يسمح لقيم الصدق للقضايا بألا تقتصر على الصفر والواحد الصحيح بل تأخذ قيما بينهما .

ومع ذلك فإن نظرية المجموعات *(Set theory)* قد تأخرت فى أن تحذو حذو هذا المنطق . فى عام ١٩٦٥ كان الأستاذ لطفى زاده *(Zadeh [43])* تقدم بالخطوة المطلوبة فقام بتعريف المجموعات الفازية *(Fuzzy sets)* فى أطروحة كلاسيكية وفكرته هى أن يسمح لقيم العضوية للعناصر المختلفة بالنسبة لكل مجموعة فازية بألا تقتصر على الصفر والواحد الصحيح بل أيضا أن تأخذ أى قيمة بينهما . وعليه فإن المجموعات الفازية كتعميم للمجموعات العادية لا تقتصر على أن تكون هى المناظر المطلوب للصفات ناقصة التحديد بل أيضا يمكن أن يعمم عليها العمليات المعتادة فى نظرية المجموعات بما يتناسب مع المعانى المتوخاة من الروابط المنطقية من عطف ونفى ولزوم وغيرها ، وبذلك وضع زاده أسس نظرية المجموعات الفازية .

على الجانب الآخر فقد بدأ الباحثون فى ايجاد تعميمات للأفكار الكلاسيكية فى موضوعات الرياضيات إلى المجموعات الفازية أمثال تلك الموضوعات التوبولوجى *(Topology)* ونظرية العدد *(Number theory)* ونظرية القياس *(Measure theory)* والجبر وحساب التفاضل.

فى عام ١٩٦٨ أدخل تشانج *(Chang [11])* مفهوم الفراغات التوبولوجية الفازية كتعميم لمفهوم الفراغات التوبولوجية بإستخدام الفئات الفازية بدلا من الفئات المعتادة .

في عام ١٩٧٩ قام كاتساراس *(Katsaras [20])* بإدخال مفهوم فراغات التقارب الفازية كتعميم لمفهوم فراغات التقارب بإستخدام الفئات الفازية بدلا من الفئات المعتادة . ثم تطورت دراسة فراغات التقارب الفازية بعد ذلك في أبحاث كثيرة منها [4, 20] .

قام الكثير من باحثي الرياضيات أمثال *Abd El-Monsef [1, 2], Azad [6], Hutton [16, 17, 18], Kerre [28, 29, 30], pu pao-Ming and Liu Ying-Ming [38, 39]*, وآخرين بتعميم الأفكار التوبولوجية إلى الحالة الفازية الأعم .

في عام ١٩٨٦ قدم بادار *(Badard [13])* الفكرة الأساسية للبناءات الملساء *(Smooth structures)* .

في عام ١٩٩٠ قدم مينج شنج-ينج *(Mingshing-Ying [32, 34])* مفهوم التوبولوجي الزغبي *(Fuzzifying topology)* .

في عام ١٩٩٢ قدم رمضان *(Ramadan [41])* دراسة عن خواص الفراغات التوبولوجية الملساء والفراغات قبل المنتظمة الملساء . كما قد أيضا الجيار *(EL-Gayyar [13])* دراسة في هذا المجال وتطبيقاته .

في عام ١٩٩٣ قدم أيضا مفهوم الفراغات المنتظمة الزغبية *(Fuzzifying uniform spaces [35])* .

في هذه الرسالة نستخدم مفهومي الفراغ التوبولوجي الزغبي والفراغ المنتظم الزغبي لتقديم ودراسة الفراغات التقاربية الزغبية *(Fuzzifying proximity spaces)* وأيضا النباءات المتجانسة الزغبية *(Fuzzifying syntopogenous structures)* . وتشمل هذه الرسالة على مقدمة يليها فصل تمهيدي ثم ثلاثة فصول تشكل صلب الرسالة ثم قائمة المراجع.

في الفصل الأول أفكار أولية عن الفئات الفازية والتقارب والانتظام والمنطق المتعدد القيم والتوبولوجي الزغبي . كما انشئت بعض الخواص .

في الفصل الثاني درسنا المفاهيم الأساسية للفراغ المنتظم الزغبي وأضفنا بعض الخواص .

في الفصل الثالث قدمنا مفهوم الفراغ التقاربي الزغبي مع دراسة خواصه ودراسة العلاقة بين الفراغ التقاربي الزغبي والفراغ المنتظم الزغبي .

فــى الفصــل الرابــع قدمنــا مفهـــوم البنـــاءات المتجانســـة الزغبيـــة *(Fuzzifying syntopogenous structures)* مع دراسة خواصه ودراســة الروابط بينـه وبيـن المفهوميـن السابقين .

ملاحظة :

تجدر الإشارة إلى أن :

١- نتائج الباب الثالث أرسلت للنشر إلى

"International Journal of Fuzzy math."

وتقبل للنشر بعد إجراء تعديلات مقترحة .

٢- نتائج الباب الرابع أرسلت للنشر إلى

"International Journal of Fuzzy math."