Fayoum University
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Curvature Symmetries in space-times and their applications in general relativity

By

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ABSTRACT

The reader will be introduced to some of the Riemannian geometry theories and concepts in the first chapter, along with terminology that will be used often throughout the dissertation, providing the foundational information required to define differentiable manifolds. More information about the fundamentals of differential geometry, specifically the idea of smooth manifolds as a useful expansion of surfaces in Euclidean space, is presented. This idea is frequently used in general relativity to build various space-times. A smooth manifold with four dimensions is used to represent space-time. We also explore semi-conformal and quasi-conformal curvature tensors on Riemannian manifolds. Additionally, we investigate singly warped product manifolds that admit semi-conformal curvature tensors. The Riemann and Ricci tensor forms of a semi-conformally flat singly warped product manifold's base and fiber manifolds are given. It is shown that a semi-conformally flat warped product manifold has constant curvature in its fiber manifold. Sufficient specifications on the warping function are provided to guarantee that the base manifold is either an Einstein manifold or a quasi-Einstein manifold. The work in the previous chapter investigates the effects on the factor manifolds of a singly warped product manifold of a flat, symmetric, or divergence-free quasi-conformal curvature tensor. Three possibilities are identified for quasi-conformally flat warped product manifolds: in one, the base manifold is quasi-Einstein, and in the other two, it has constant curvature. Conversely, the fiber manifold is Einstein in one case and shows constant curvature in two others. Three different cases arise for quasi-conformally symmetric warped product manifolds: the first case is when the base manifold is Cartan symmetric, and the fiber has constant curvature; the second case is when the base

manifold is Cartan symmetric and the fiber is Einstein; and the third case is when the fiber is Cartan symmetric and the Ricci tensor of the base manifold is of Codazzi type. Finally, the study gives constraints on singly warped product manifolds whose factor manifolds contain harmonic Riemann curvature tensors and a divergence-free quasi-conformal curvature tensor.

The thesis consists of four chapters arranged as follows:

Chapter 1: The first chapter aims to familiarize the reader with some of the concepts and theories of Riemannian geometry, as well as some of the terminology that is frequently used in the dissertation. We first provide the background information needed for the definition of differentiable manifolds. For additional details regarding the foundations of differential geometry, we present the concept of smooth manifolds as a helpful extension of surfaces in Euclidean space. This idea is frequently used in general relativity to build various space-times. A smooth manifold with four dimensions is used to represent a space-time.

Chapter 2: In the second chapter, we investigate singly warped product manifolds. In differential geometry and mathematical physics, notably in general relativity, warped product manifolds play a significant role. This concept is a helpful extension of a manifold with Riemannian products.

Chapter 3: Singly warped product manifolds admitting semi-conformal curvature tensors are the subject of this study. The Riemann and Ricci tensor forms of a semi-conformally flat singly warped product manifold's base and fiber manifolds are provided. It is shown that a semi-conformally flat warped product manifold has constant curvature in its fiber manifold. Sufficient specifications on

the warping function are given to guarantee that the base manifold is either an Einstein manifold or a quasi-Einstein manifold.

Chapter 4: This chapter studies the consequences of the existence of a flat, symmetric, or divergence-free quasi-conformal curvature tensor on the factor manifolds of a singly warped product manifold. Three unique scenarios are seen in quasi-conformally flat warped product manifolds: in one, the base manifold is quasi-Einstein, and in the other two, it has constant curvature. Alternatively, the fiber manifold is Einstein in one case and has constant curvature in two others. Three distinct cases are found in quasi-conformally symmetric warped product manifolds: the first case involves a base manifold that is Ricci symmetric and a fiber that is Einstein; the second case involves a base manifold that is Cartan symmetric and a fiber that has constant curvature; and the final case involves a Cartan symmetric fiber and a Ricci tensor of Codazzi type. Lastly, requirements to guarantee the harmonicity of the Riemann curvature tensors of the factor manifolds are given for singly warped product manifolds that admit a quasi-conformal curvature tensor without divergence.