Fayoum University
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Studies on Some Classes of Analytic Functions Associated with Certain Operators

A Thesis

Submitted in the Partial Fulfillment of the Requirements for the Degree of Master of Science

In
Pure Mathematics
(Complex Analysis)

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2015

Summary

The purpose of this thesis is to define and study properies for certain classes of univalent and $p-$valent functions defined in the open unit disc $U=\{z\in\mathbb{C}:|z|<1\}$, where $\mathbb{C}$ is the complex plane. These classes are defined by using some linear operators, integral operators, Hadamard product (or convolution) and higher order derivative .

Let A denote the class of all functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k,$$

which are analytic in U . Also let S denote the subclass of functions of A which are univalent in U. Further let T the subclass of S all functions of the form:

$$f(z)=z-\sum_{k=2}^{\infty}a_k z^k\ (a_k\geq 0)$$

For two functions $f(z)$ and $g(z)$ in A given by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$ the Hadamard product (or convolution) $(f * g)(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z)$$

Definition 1 [70]**.** A function $f(z) \in S$ is said to be starlike of order α if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$$

for some α $(0 \le \alpha < 1)$ and for all $z \in U$. The class of all starlike functions of order α is denoted by $S^*(\alpha)$

Definition 2 [70] . A function $f(z)$ belonging to S is said to be convex of order α if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha$$

for some α $(0 \le \alpha < 1)$ and for all $z \in U$. The class of all convex functions of order α is denoted by $K(\alpha)$.

The classes $S^*(\alpha)$ and $K(\alpha)$ were studied subsequently by Schild, [74], MacGregor, [53], Jack, [38], Pinchuk, [68] and others.

One can see that

$$f(z) \in K(\alpha) \Leftrightarrow zf'(z) \in S^*(\alpha)$$

Definition 3 [34]. A function $f(z) \in S$ is said to be close-to-convex of order $\beta\ (0 \le \beta < 1)$, if there exist a function $g(z) \in S^*$ such that

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \beta, \quad z \in U.$$

We denote by $C_g(\beta)$ the class of all close-to-convex functions of order β (see Goodman [34]).
Also, we note that:

$$K \subset S^* \subset C_g(\beta) \subset C_g \subset S,$$

where C_g is the class of all close-to-convex functions (see Kaplan [44]).
We note that:

$$T^*(\alpha) = S^*(\alpha) \cap T \text{ and } C(\alpha) = K(\alpha) \cap T.$$

Goodman, [35] introduced and defined the following subclasses of K and S^*. A function $f(z) \in A$ is said to be uniformly convex (uniformly starlike) in U if $f(z)$ is in K (S^*) and has the property that for every circular arc γ contained in U, with center ζ also in U, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\zeta)$. The class of uniformly convex (starlike) functions is denoted by UCV and UST, respectively.

Definition 4 ([35],[52] and [71]). A function $f(z) \in A$ is said to be in the class of uniformly convex functions, UCV, if it satisfies the following condition:

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in U)$$

Further, a function $f(z) \in A$ is said to be in the class of uniformly starlike functions, UST, if it satisfies the following condition:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)}-1\right| \quad (z \in U)$$

The class UCV was introduced by Goodman [35] and Ma and Minda [52]. The class UST was introducing by Goodman, [36] and Ronning, [72]. One can see that

$$f(z) \in UCV \Leftrightarrow zf'(z) \in UST.$$

In [72], Ronning generalized the classes UST and UCV by introducing a parameter $\alpha\,(-1 \leq \alpha < 1)$ in the following way.

Definition 5 [72]**.** A function $f(z) \in A$ is said to be in the class of uniformly starlike functions of order α, $UST(\alpha)$, if itsatisfies the following condition:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\} \geq \left|\frac{zf'(z)}{f(z)}-1\right| \quad (-1 \leq \alpha < 1; z \in U)$$

Replacing $f(z)$ in (3) by $zf'(z)$ we have the condition:

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}-\alpha\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right| \quad (-1 \leq \alpha \leq 1;\, z \in \mathcal{U})$$

required for the function $f(z)$ to be in the class $UCV(\alpha)$ of uniformly convex functions of order α. One can see that

$$f(z) \in UCV(\alpha) \Leftrightarrow zf'(z) \in UST(\alpha)$$

Kanas and Wisniowska [42] and [43] introduced the classes of k-uniformly convex functions $k-UCV$ $(0 \leq k < \infty)$ and k-uniformly starlike functions $k-UCV$ $(0 \leq k < \infty)$, as follows:

Definition 6 ([42] and [43]). A function $f(z) \in \mathcal{A}$ is said to be in the class of $k-$ uniformly convex functions, $k-UCV$, if it satisfies the following condition:

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq k\left|\frac{zf''(z)}{f'(z)}\right| \quad (k \geq 0;\, z \in \mathcal{U})$$

From (2), we can easily see that the class $k-UST$, of k-uniformly starlike functions is associated with $k-UCV$ by the relation

$$f(z) \in k-UCV \Leftrightarrow zf'(z) \in k-UST.$$

Thus, the class $k-UST$, is the subclass of $\mathcal{A}$ satisfies the following condition:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq k\left|\frac{zf'(z)}{f(z)}-1\right| \quad (k \geq 0;\, z \in \mathcal{U})$$

Definition 7 ([76], [62] and [10]). A function $f(z) \in A$ is said to be in the class of uniformly starlike functions of order α and type β, $UST(\alpha,\beta)\ (-1 \le \alpha < 1, \beta \ge 0)$, if it satisfies the following condition:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \ge \beta\left|\frac{zf'(z)}{f(z)} - 1\right| \quad (-1 \le \alpha < 1; \beta \ge 0; z \in U)$$

Definition 8 ([62] and [10]). A function $f(z) \in A$ is said to be in the class of uniformly convex functions of order α and type β, $UCV(\alpha,\beta)\ (-1 \le \alpha < 1, \beta \ge 0)$, if it satisfies the following condition:

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \ge \beta\left|\frac{zf''(z)}{f'(z)}\right| \quad (-1 \le \alpha < 1; \beta \ge 0; z \in U)$$

From (4) and (5), we have

$$f(z) \in UCV(\alpha,\beta) \Leftrightarrow zf'(z) \in UST(\alpha,\beta)$$

We note that:

$$(i)\ UST(0,1) \equiv UST \text{ and } UST(\alpha,1) \equiv UST(\alpha),$$

$$(ii)\ UCV(0,1) \equiv UCV \text{ and } UCV(\alpha,1) \equiv UCV(\alpha).$$

For complex or positive real parameters $\alpha_1,\dots,\alpha_q$ and $\beta_1,\dots,\beta_s$

$\left(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-, \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}; j = 1, 2, \ldots, s\right)$ the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_s; z)$ is defined by

$$ {}_qF_s(\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k}{(\beta_1)_k \ldots (\beta_s)_k} \frac{1}{k!} z^k, $$

$$ \left(q \leq s+1; s, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\}; z \in \mathbb{U}\right), $$

where $(\theta)_k$ is the Pochhammer symbol defined in terms of the Gamma function Γ by

$$ (\theta)_k = \frac{\Gamma(\theta + k)}{\Gamma(\theta)} = \begin{cases} 1 & (k = 0) \\ \theta(\theta+1)\ldots(\theta+k-1) & (k \in \mathbb{N}) \end{cases} $$

Using the function

$$ h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z\,{}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z), $$

Dziok and Srivastava (see [29]) defined the linear operator $H_{q,s}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) : \mathcal{A} \to \mathcal{A}$, by

$$ H_{q,s}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)f(z) = h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z) $$

$$ = z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1) a_k z^k \quad (z \in \mathbb{U}), $$

where

$$ \Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1} \ldots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \ldots (\beta_s)_{k-1}} \frac{1}{(k-1)!}. $$

For brevity, we write

$$ H_{q,s}(\alpha_1)f(z) = H_{q,s}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)f(z) $$

Specializing the parameters $\alpha_1, \alpha_2, \beta_1, q$ and s, we obtain many linear operators studied by various authors (see Carlson and Shaffer [21], Hohlov [37], Ruscheweyh [73], Owa and Srivastava [66], Choi et al. [28], Noor [58], Cho et al. [24], Bernardi [18] and others).

Jung et al. [40] introduced the following one-parameter families integral operators

$$Q^{\alpha}_{\beta} f(z) = \begin{cases} \binom{\alpha+\beta}{\beta} \dfrac{\alpha}{z^{\beta}} \displaystyle\int_0^z \left(1-\frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\,dt & (\alpha > 0;\ \beta > -1) \\ f(z) & (\alpha = 0;\ \beta > -1) \end{cases}$$

and

$$I^{\alpha} f(z) = \begin{cases} \dfrac{2^{\alpha}}{z\Gamma(\alpha)} \displaystyle\int_0^z \left(\log\frac{z}{t}\right)^{\alpha-1} f(t)\,dt & (\alpha > 0) \\ f(z) & (\alpha = 0) \end{cases}.$$

For $f(z) \in \mathcal{A}$ given by (1), we deduce that

$$Q^{\alpha}_{\beta} f(z) = z + \frac{\Gamma(\beta+\alpha+1)}{\Gamma(\beta+1)} \sum_{k=2}^{\infty} \frac{\Gamma(\beta+k)}{\Gamma(\beta+\alpha+k)} a_k z^k \quad (\alpha \geq 0;\ \beta > -1)$$

and

$$I^{\alpha} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1}\right)^{\alpha} a_k z^k \quad (\alpha \geq 0).$$

From (7) and (8), it easily to verify the following identities:

$$z\left(Q^{\alpha+1}_{\beta} f(z)\right)' = (\alpha+\beta+1) Q^{\alpha}_{\beta} f(z) - (\alpha+\beta) Q^{\alpha+1}_{\beta} f(z) \quad (\alpha > 0;\ \beta > -1),$$

and

$$z\left(I^{\alpha+1} f(z)\right)' = 2I^{\alpha} f(z) - I^{\alpha+1} f(z) \quad (\alpha > 0)$$

Putting $\beta = v > -1, \alpha = 1,$ we note that

$$Q_v^1 f(z) = J_v f(z) = \frac{v+1}{z^v} \int_0^z t^{v-1} f(t)\,dt$$

$$= z + \sum_{k=2}^{\infty} \frac{v+1}{v+k} a_k z^k \quad (v > -1; z \in U),$$

where J_v is the Bernardi-libera-livingston integral operator (see, [28], [32] and [65]).

Let $S(p)$ denote the class of $p-$valent functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}),$$

and $S(p,n)$ denote the class of $p-$valent functions of the form:

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (p \in \mathbb{N}),$$

which are analytic and $p-$valent in U. We note that $S(p,1) = S(p)$ and $S(1,1) = S.$

Let $T(p)$ denote the class of all functions of the form:

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \quad (a_k \geq 0; p \in \mathbb{N}),$$

which are analytic and $p-$valent in U. We note that $T(1) = T.$

Let $T(p,n)$ denote the subclass of $S(p,n)$ of functions of the form:

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \quad (a_k \geq 0; p \in \mathbb{N}),$$

We note that $T(p,1) = T(p)$.

Let Σ denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n ,$$

which are analytic in $U^* = U \setminus \{0\}$. If $g(z) \in \Sigma$, be given by

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n ,$$

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n = (g * f)(z)$$

Denote by Σ_p the class of analytic and univalent functions in the punctured disc U^* of the form:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (a_n \geq 0) .$$

Let $\Sigma(p)$ denote the class of meromorphic $p-$ valent functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (p \in \mathrm{N}),$$

which are analytic in U^*. For $p = 1$ we have $\Sigma(1) = \Sigma$.

Let $\Sigma_p(p)$ be the class of missing functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=p}^{\infty} a_k z^k \quad (z \in U^*)$$

Using the generalized hypergeometric function ${}_qF_s(\alpha_1,\ldots,\alpha_q,\beta_1,\ldots,\beta_s;z)$ defined by (6),
Liu and Srivastava [50] (see also Aouf [5]) defined the operator $M_{p,q,s}(\alpha_1)$ as follows:

$$m_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)=z^p\,{}_qF_s(\alpha_1,\ldots,\alpha_q,\beta_1,\ldots,\beta_s;z)$$

consider the linear operator

$$M_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s):\mathcal{A}_p(p)\to\mathcal{A}_p(p),$$

which is defined by means of the following Hadamard product (or convolution):

$$M_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)=m_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)*f(z)$$

For a function $f(z)\in\mathcal{A}_p(p)$, we have

$$M_{p,q,s}(\alpha_1)=M_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)=z^p+\sum_{k=p}^{\infty}\Gamma_{k-p}(\alpha_1)\,a_k\,z^k$$

$$(q\le s+1;\ q,s\in\mathbb{N}_0;\ z\in U),$$

where, for convenience,

$$\Gamma_{k-p}(\alpha_1)=\frac{(\alpha_1)_{k-p}\ldots(\alpha_q)_{k-p}}{(k-p)!(\beta_1)_{k-p}\ldots(\beta_s)_{k-p}}\quad(k\in\mathbb{N}).$$

This thesis consists of five Chapters.

Chapter 1

This chapter is considered as an introductory chapter and consists of six sections:

In Section 1.1, some basic concepts of univalent functions are introduced.

In Section 1.2, definitions of uniformly convex (starlike) functions are introduced.

In Section 1.3, some linear operators associated with analytic functions are defined.

In Section 1.4, basic concepts of $p-$ valent functions are introduced.

In Section 1.5, basic concepts of meromorphic univalent functions are introduced and contains the definitions of the classes $\Sigma(\alpha,\beta)$ and $\Sigma_p(\alpha,\beta)$.

Definition 9. [45]**.** For $0 \le \alpha < 1$ and $0 \le \beta < 1$, let $\Sigma(\alpha,\beta)$ denote a subclass of Σ consisting of functions of the form (12) satisfying the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{(\beta-1)f(z)+\beta zf'(z)}\right) > \alpha \quad (z \in U^*).$$

Furthermore, we say that a function $f \in \Sigma_p(\alpha,\beta)$ whenever $f(z)$ is of the form (13) and satisfying (16) .

In Section 1.6, some basic concepts of $p-$ valent meromorphic functions are introduced.

Chapter 2

This chapter consists of seven sections. The first section is an introductory section and contains the definitions of the classes $S_n(p,q,\alpha)$ and $C_n(p,q,\alpha)$.

For function $f(z)$ defined by (11), we define the classes $S_n(p,q,\alpha)$ and $C_n(p,q,\alpha)$ as follows:

$$S_n(p,q,\alpha)=\left\{ f\in T(p,n):\ \mathrm{Re}\left(\frac{z f^{(1+q)}(z)}{f^{(q)}(z)}\right)>\alpha\ (z\in U)\right\},$$

and

$$C_n(p,q,\alpha)=\left\{ f\in T(p,n):\ \mathrm{Re}\left(1+\frac{z f^{(2+q)}(z)}{f^{(1+q)}(z)}\right)>\alpha\ (z\in U)\right\},$$

where, for each $f\in T(p,n)$, we have

$$f^{(q)}(z)=\delta(p,q)z^{p-q}-\sum_{k=n+p}^{\infty}\delta(k,q)a_k z^{k-q},$$

and

$$\delta(p,j)=\frac{p!}{(p-j)!}=\begin{cases}1 & (j=0)\\ p(p-1)\ldots(p-j+1) & (j\neq 0)\end{cases}.$$

In Section 2.2, contains the definition of the class $TC_m(p,q,n,\alpha)\ (m\in N_0)$ as follows:

Definition 10. A function $f(z)$ defined by (11) and belonging to the class $T(p,n)$ is said to be in the class $TC_m(p,q,n,\alpha)$ if it also satisfies the coefficient inequality:

$$\sum_{k=n+p}^{\infty}\left(\frac{k+q}{p+q}\right)^{m}[k+q+\alpha(k,q)]a_k \le [p+q+\alpha(p,q)]$$

In Section 2.3, growth and distortion theorems for functions in the class $TC_m(p,q,n,\alpha)$ are obtained.

In Section 2.4, closure theorems for functions in the class $TC_m(p,q,n,\alpha)$ are obtained.

In Section 2.5, extreme points for functions in the class $TC_m(p,q,n,\alpha)$ are obtained.

In Section 2.6, modified Hadamard product for functions in the class $TC_m(p,q,n,\alpha)$ are obtained.

In Section 2.7, radii of close-to-convexity, starlikeness and convexity for functions in the class $TC_m(p,q,n,\alpha)$ are obtained.

Chapter 3

This chapter consists of two sections. The first section is an introductory section and contains the definitions of the classes $ST_n(\beta,\alpha)$, $CT_n(\beta,\alpha)$ and $UL_n(\beta,\alpha,\gamma)$ and the definition of Hölder inequality.

Let $T(n)$ denote the class of analytic functions in U of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad \left(a_k \ge 0\,;\, n \in \mathbb{N} = \{1,2,3,\ldots\}\right).$$

Also we define the classes $ST_n(\beta,\alpha)$ and $CT_n(\beta,\alpha)$ as follows:

$$\mathrm{ST}_n(\alpha,\beta)=\left\{\begin{array}{c} f\in \mathrm{T}(n):\ \mathrm{Re}\left\{\dfrac{zf'(z)}{f(z)}-\alpha\right\}\geq \beta\left|\dfrac{zf'(z)}{f(z)}-1\right| \\ (0\leq\alpha<1;\ \beta\geq 0;\ z\in \mathrm{U}) \end{array}\right\},$$

and

$$\mathrm{CT}_n(\alpha,\beta)=\left\{\begin{array}{c} f\in \mathrm{T}(n):\ \mathrm{Re}\left\{1+\dfrac{zf''(z)}{f'(z)}-\alpha\right\}\geq \beta\left|\dfrac{zf''(z)}{f'(z)}\right| \\ (0\leq\alpha<1;\ \beta\geq 0;\ z\in \mathrm{U}) \end{array}\right\}.$$

Definition 11 [16]**.** For $0\leq\alpha<1$, $\beta\geq 0$ and $0\leq\gamma\leq 1$, a function $f\in \mathrm{T}(n)$ is said to be in the subclass $\mathrm{U}L_n(\alpha,\beta,\gamma)$ of $\mathrm{T}(n)$ if the following inequality holds:

$$\mathrm{Re}\left\{\frac{zf'(z)+\gamma z^2 f''(z)}{(1-\gamma)f(z)+\gamma zf'(z)}-\alpha\right\}\geq \beta\left|\frac{zf'(z)+\gamma z^2 f''(z)}{(1-\gamma)f(z)+\gamma zf'(z)}-1\right|.$$

Definition 12 [17]. For $p_i\geq 1$ and $\sum_{i=1}^{m}\frac{1}{p_i}=1,$ the Hölder inequality is defined by:

$$\sum_{i=1}^{n}\left(\prod_{j=1}^{m}a_{i,j}\right)\leq\prod_{j=1}^{m}\left(\sum_{i=1}^{n}a_{i,j}^{p_i}\right)^{\frac{1}{p_i}}.$$

In Section 3.2, main results for functions in the classes $\mathrm{ST}_n(\alpha,\beta)$, $\mathrm{CT}_n(\alpha,\beta)$ and $\mathrm{U}L_n(\alpha,\beta,\gamma)$ are obtained.

Chapter 4

This chapter consists of two sections.The first section is an introductory section and contains the definitions of the classes $\mathcal{R}$ and $\mathcal{T}$

Definition 13. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{R}$ if it satisfies inquelity

$$\operatorname{Re}\left(\frac{Q f(z)}{Q f(z)}\right) \quad 0; \quad 1; 0 \quad 1; z \in \mathcal{U}$$

Definition 14. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{T}$ if it satisfies inquelity

$$\operatorname{Re}\left\{\frac{I f(z)}{I f(z)}\right\} \quad \frac{1}{2} \quad \left(z \in \mathcal{U};\ 0 \quad 1\right).$$

In Section 4.2, some inclusion relations of the classes $\mathcal{R}$ and $\mathcal{T}$ are obtained.

Chapter 5

This chapter consists of five sections. The first section is an introductory section and contains the definition of the classes $\Sigma^*_{p,q,s}(\alpha_1;A,B,\beta)$ and $\Sigma^*_{p,q,s,c}(\alpha_1;A,B,\beta)$ as follow:

Definition 15 [8]. For a function $f(z) \in \Sigma_p(p)$, we say that $f(z)$ is in the class $\Sigma_{p,q,s}(\alpha_1;A,B,\beta)$ of meromorphically $p-$ valent functions in U if and only if

$$\left| \frac{\dfrac{z\left(M_{p,q,s}(\alpha_1)f(z)\right)'}{M_{p,q,s}(\alpha_1)f(z)}+p}{B\dfrac{z\left(M_{p,q,s}(\alpha_1)f(z)\right)'}{M_{p,q,s}(\alpha_1)f(z)}+\left[pB+(A-B)(p-\beta)\right]} \right| < 1$$

$$(-1 \le B < A \le 1;\, 0 \le \beta < p;\, p \in N,\, z \in U).$$

Let $\Sigma^*_p(p)$ be the subclass of $\Sigma_p(p)$ consisting of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=p}^{\infty} |a_k|\, z^k \quad (p \in N).$$

Also let $\Sigma^*_{p,q,s}(\alpha_1;A,B,\beta)$ be the subclass of $\Sigma_{p,q,s}(\alpha_1;A,B,\beta)$ such that

$$\Sigma^*_{p,q,s}(\alpha_1;A,B,\beta) = \Sigma_{p,q,s}(\alpha_1;A,B,\beta) \cap \Sigma^*_p(p).$$

The classes $\Sigma_{p,q,s}(\alpha_1;A,B,\beta)$ and $\Sigma^*_{p,q,s}(\alpha_1;A,B,\beta)$ were introduced and studied by Aouf [8].

Definition 16. Let $\Sigma^*_{p,q,s,c}(\alpha_1;A,B,\beta)$ denote the subclass of $\Sigma^*_{p,q,s}(\alpha_1;A,B,\beta)$ consisting of functions of the form:

$$f(z) = \frac{1}{z^p} + \frac{(A-B)(p-\beta)}{(p(1-B)-(A-B)(p-\beta))\Gamma_p(\alpha_1)} z^p + \sum_{k=p+1}^{\infty} |a_k| z^k \quad (0 < c < 1)$$

In Section 5.2, properties for functions in the class $\Sigma_{p,q,s,c}(\alpha_1; A, B, \beta)$ are obtained.

In Section 5.3, closure theorems for functions in the class $\Sigma_{p,q,s,c}(\alpha_1; A, B, \beta)$ are obtained.

In Section 5.4, radius of convexity for functions in the class $\Sigma_{p,q,s,c}(\alpha_1; A, B, \beta)$ are obtained.

In Section 5.5, applying the technique used by Silverman [78], we investigate the ratio of a function $f(z) \in \Sigma_p$ to its sequence of partial sums $f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n$, when the coefficients of $f(z)$ are sufficiently small to satisfy the condition $f(z) \in \Sigma_p(\alpha, \beta)$. More precisely, we determine sharp lower bounds for $\operatorname{Re}\left\{\frac{f(z)}{f_k(z)}\right\}$, $\operatorname{Re}\left\{\frac{f_k(z)}{f(z)}\right\}$, $\operatorname{Re}\left\{\frac{f'(z)}{f_k'(z)}\right\}$, and $\operatorname{Re}\left\{\frac{f_k'(z)}{f'(z)}\right\}$.