<u>Chapter 4</u>
<u>Superposition and Standing Waves</u>

<u>Lesson 10:</u>

Objectives:

The student will be able to:

- Define the superposition & the interference
- Define the types of the interference
- Define the standing wave.
- Describe the formation of standing waves.
- Describe the characteristics of standing waves.
- Define the resonance phenomena.
- Define the standing wave in air columns.

***Key words:**

- **Standing wave**: superposition of two identical waves propagating in opposite directions.
- **Nodes** : the points of zero amplitude,
- **Antinodes:** the points of max amplitudes, where constructive interference is greatest.
- **Loops:** the regions of greatest amplitude in a standing wave

4.1 Superposition and Interference:

Since many wave phenomena cannot be described by a single traveling wave. We must analyze <u>complex waves in terms of a combination of traveling waves</u>. So we can use the superposition principle which said that

"If two or more traveling waves are moving through a medium, the resultant value of the wave function at any point is the algebraic sum of the values of the wave functions of the individual waves"

Linear Waves:

The waves which obey the superposition principle, as we see in opposite figure, the wave function for the pulse moving to the right is y_1, and that for the pulse moving to the left is y_2 (fig4.1a). The pulses have the same speed but different shapes, and the displacement of the elements of the medium is in the positive y direction. When the waves go to overlap (fig4.1b), the wave function for the resulting wave is y_1+y_2. When the crests of the pulses coincide (fig4.1c), the resulting wave has larger amplitude than that of the individual pulses. Finally, the two pulses separate and continue moving in their original direction (fig4.1d).

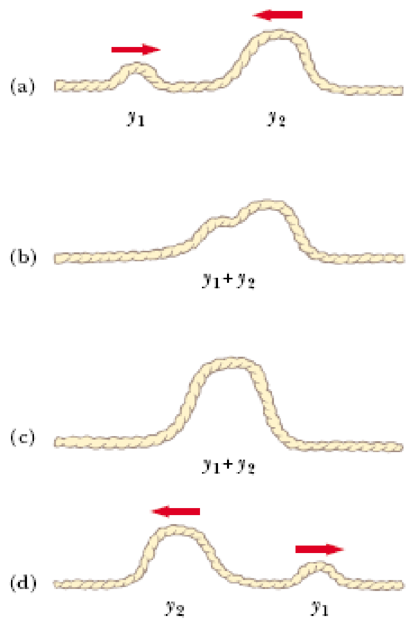
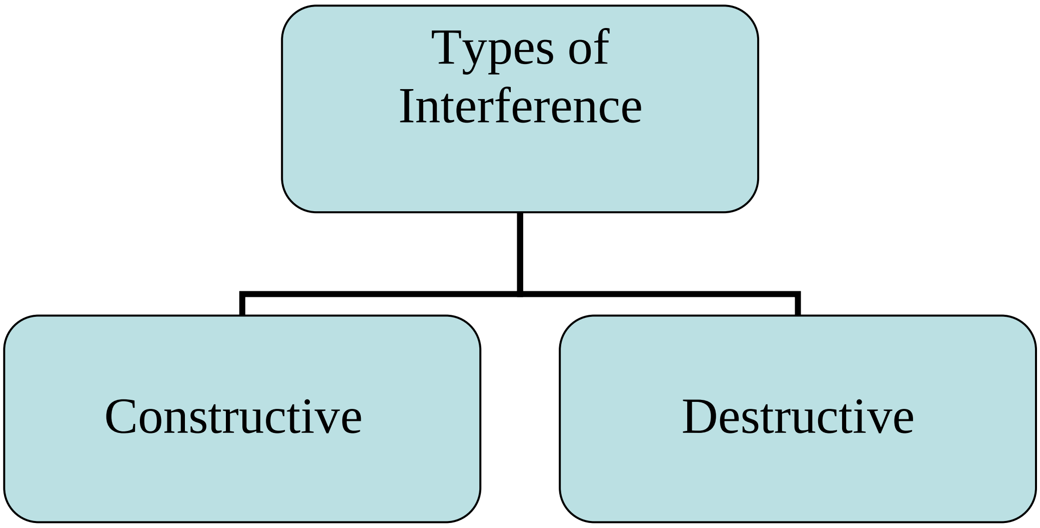


Figure 4.1: a-d two plus traveling on a stretched string in opposite direction.
b, c the net displacement of the string equals the sum of the displacements of the string.

Interference:

Defined as the combination of separate waves in the region of space, and the resultant pulse has amplitude greater than that of their individual pulse.



a- Constructive Interference

Formed when the displacements caused by the two pulses are in the same direction.

b- Destructive Interference:

Formed when the displacements caused by the two pulses are in opposite direction.

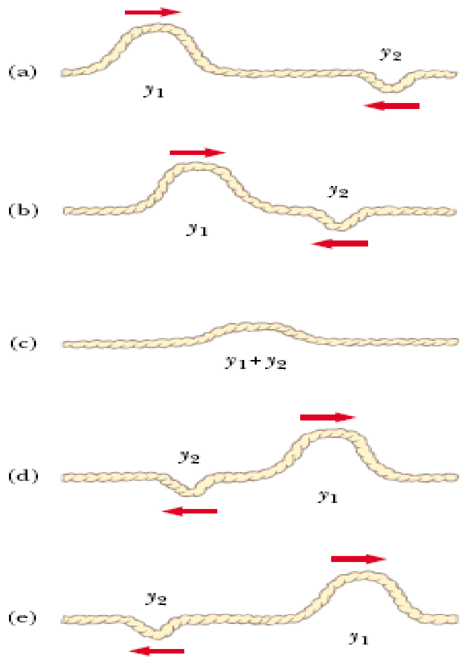


Figure 4.2:
Two pulses traveling in opposite direction and having displacements that are inverted relative to each other. When the two overlap in (c), their displacements cancel each other.

4.2 Superposition of Sinusoidal Waves:

At this time we apply the principle of superposition to two sinusoidal waves traveling in the same direction. If the two waves have the same frequency, wavelength, and amplitude but differ in phase as we can express

$$y_1 = A \sin (kx-\omega t) \qquad \& \qquad y_2 = A \sin (kx-\omega t+\Phi)$$

Where $k=2\pi/\lambda$, $\omega = 2\pi f$, and Φ is the phase constant.

The resultant wave function is

$$Y = y_1 + y_2 = A[\sin (kx-\omega t) + \sin (kx-\omega t+\Phi)]$$

Since $\sin a + \sin b = 2 \cos [(a-b)/2] \sin [(a+b)/2]$

We find that

$$Y = 2A \cos (\Phi/2) \sin (kx-\omega t+\Phi/2)$$

From this result we conclude that

- The resultant wave function Y is sinusoidal and has the same frequency and wavelength as the individual waves.
- The amplitude of the resultant wave is $2A \cos (\Phi/2)$ and its phase is $\Phi/2$.

If the phase constant Φ equals zero, then $\cos (\Phi/2) = \cos 0 = 1$, and the amplitude of the resultant wave is 2A. in this case the waves are said to be in phase and thus interference constructively. As shown in fig 4.3 a

Generally, constructive interference occurs when cos (Φ/2) = ±1 this is when Φ = 0, 2π, 4π, ……….rad
But when Φ is equal to π rad or any odd multiple of π, then cos (Φ/2) = cos (π /2) = 0, and the crests of one wave occur at the same positions as the troughs of the second wave so the resultant wave has zero amplitude.

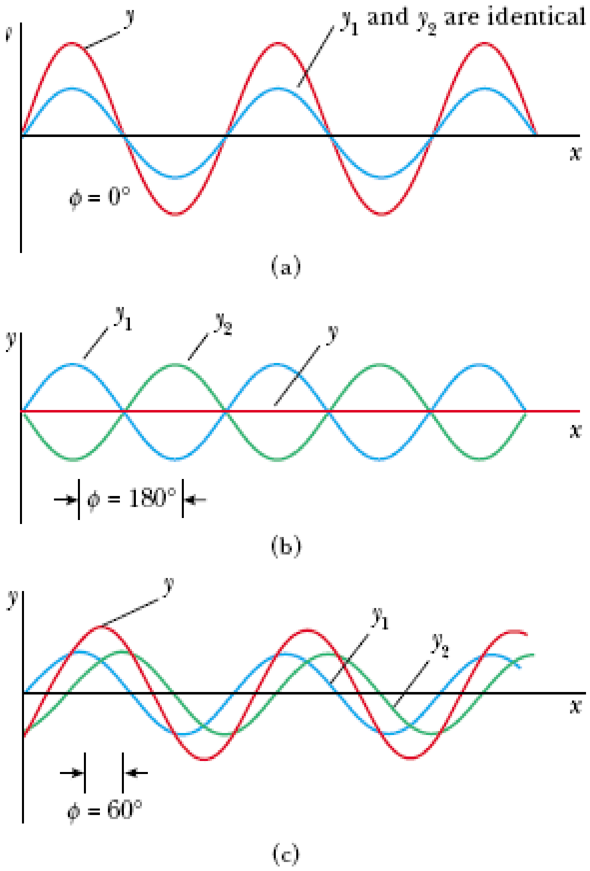


Figure 4.3 the superposition of two identical waves y_1 and y_2 (blue and green) to yield a resultant wave (red).
(a) When y_1 and y_2 are in phase, the result is constructive interference.
(b) When y_1 and y_2 are π rad out of phase, the result is destructive interference.
(c) When the phase angle has a value other than 0 or π rad, the resultant wave Y falls somewhere between the shown in (a) and (b)

4.3 Standing Waves:

In your opinion, what will happen when two speakers face each other and each one emits sound waves of the same frequency and amplitude????
Come with me to analyze this situation. Consider two transverse sinusoidal waves having the same frequency, amplitude, and wavelength but traveling in opposite directions

y_1= A sin (kx-ωt) traveling in +X

y_2= A sin (kx+ωt)   traveling in –X

$$Y = y_1 + y_2 = A \sin(kx-\omega t) + A \sin(kx+\omega t) \qquad (4.1)$$

Since sin(a ± b) = sin(a) cos(b) ± cos(a) sin(b)

So equation (4-1) reduces to

$$Y = (2A \sin kx) \cos \omega t \qquad (4.2)$$

Equation (4.2) represents the wave function of a standing wave which defined as the superposition of two identical waves traveling in opposite directions as we see in Fig(4-5).

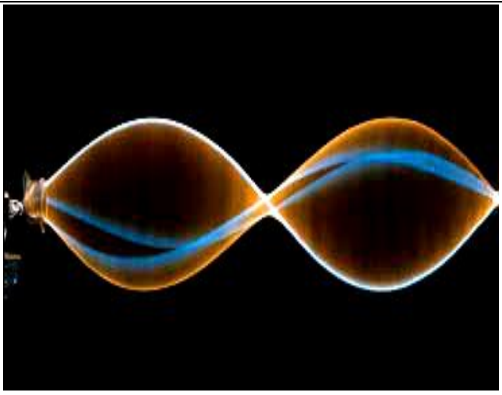


Figure 4.5 Multiflash photograph of a standing wave on a string. The time behavior of the vertical displacement from equilibrium of an individual element of the string is given by cos ωt. The amplitude of the vertical oscillation of any elements of the string depends on the horizontal position of the element. Each element vibrates within the confines of the envelope function 2A sin kx.

As we see equation (4.2) describes a special kind of simple harmonic motion. Each element of the medium oscillates in simple harmonic motion with the frequency ω, the amplitude of the simple harmonic motion of a given element is 2Asin kx.

The maximum amplitude of an element of the medium has a minimum value of zero when x satisfies the condition sin kx = 0

OR **when kx = π, 2π, 3π…….. Since k = (2π/ λ) ⇨x = λ/2, λ, 3λ/2**

So we have two definitions

Nodes:

Are the points of zero amplitude.

Antinodes:

Are the position in the medium at which the maximum amplitude occurs, so antinodes are located at positions for which x satisfies the condition sin kx = ± 1

OR when kx = π/2, 3π/2, 5π/2,........Since k = (2π/ λ) ⇨x = λ/4, 3λ/4, 5λ/4……

Wave patterns of the elements of the medium produced at various times by two waves traveling in opposite directions are shown in next Figure. The blue and green curves are the wave patterns for the individual traveling waves, and the red curves are the wave patterns for the resultant standing wave.

At t =0 (Fig. 4.6a), the two traveling waves are in phase, giving a wave pattern in which each element of the medium is experiencing its maximum displacement from equilibrium.

At t =T/4 (Fig. 4.6b), the traveling waves have moved one quarter of a wavelength (one to the right and the other to the left). At this time, the traveling waves are out of phase, and each element of the medium is passing through the equilibrium position in its simple harmonic motion. The result is zero displacement for elements at all values of x—that is, the wave pattern is a straight line.
At t = T/2 (Fig. 4.6c), the traveling waves are again in phase, producing a wave pattern that is inverted relative to the t = 0 pattern. In the standing wave, the elements of the medium alternate in time between the extremes shown in Figure (a&c).

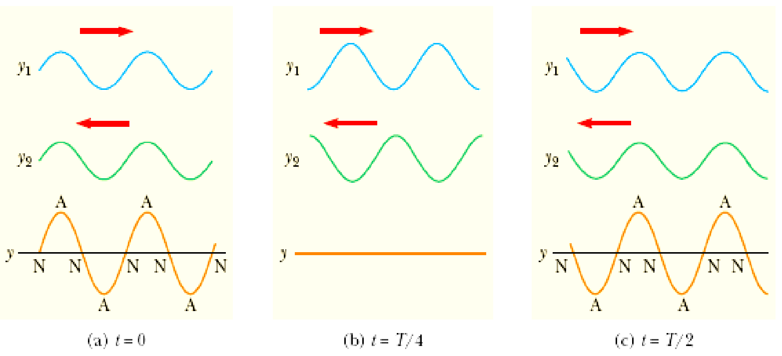


Figure 4.6
Standing wave patterns produced at various times by two waves of equal amplitude

Example 4.1 Formation of a Standing Wave

Two waves traveling in opposite directions produced a standing wave. The individual wave functions are

$y_1 = 4 \sin(0.3x - 2t)$, $y_2 = 4 \sin(0.3x + 2t)$ **where** x and y are measured in centimeters.

(A) Find the amplitude of the simple harmonic motion of the element of the medium located at x = 2.3 cm

(B) Find the position of the nodes and antinodes if one end of the string is at x = 0

(C) What is the maximum value of the position in the simple harmonic motion of an element located at an antinode ?

Solution(A):

The standing wave is described by equation (4.2); in our problem, we have A = 4 cm, k = 3 rad/cm, and ω = 2 rad/s. thus,

$y = (2A \sin kx) \cos \omega t = (8 \sin 3x) \cos 2t$

Thus, we obtain the amplitude of the simple harmonic motion of the element at the position x =2.3cm by evaluating the coefficient of the cosine function at this position:

$$y_{max} = 8 \sin 3x \,(x = 2.3) = 8 \sin 6.9 = 4.6 \text{ cm}$$

Solution(B):

With $k = (2\pi/ \lambda) = 3$ rad/cm, and the wavelength is $\lambda = (2\pi/ 3)$.

So we find the nodes are located at:

$$x = (n\lambda/2) = n\pi/3 \qquad n = 0,1,2,3,\ldots$$

also we find that the antinodes are located at:

$$x = (n\lambda/4) = n\pi/6 \qquad n = 1,3,5,\ldots$$

Solution(C):

The maximum position of an element at an antinode is the amplitude of the standing wave, which is twice the amplitude of the individual traveling waves:

$$y_{max} = 2A (\sin kx)_{max} = 2(4.0)((1) = \pm 8 \text{ cm}$$

where we have used the fact that the maximum value of sin kx is ±1. Let us check this result by evaluating the coefficient of our standing-wave function at the positions we found for the antinodes:

$$y_{max} = 8 \sin 3x \,(x = (n\pi/6) = 8 \sin (n\pi/2) = \pm 8\text{cm}$$

4.4 Standing Waves in String Fixed at Both Ends:

Consider a string of length L fixed at both ends. Standing waves are set up in the string by a continuous superposition of waves incident on and reflected from the ends.

Note:

There is a boundary condition for the waves on the string. There is a node at ends of the string because they are fixed, must have zero displacement.

The boundary condition results in the string having a number of natural patterns of oscillation, called normal modes, each of which has a characteristic frequency.

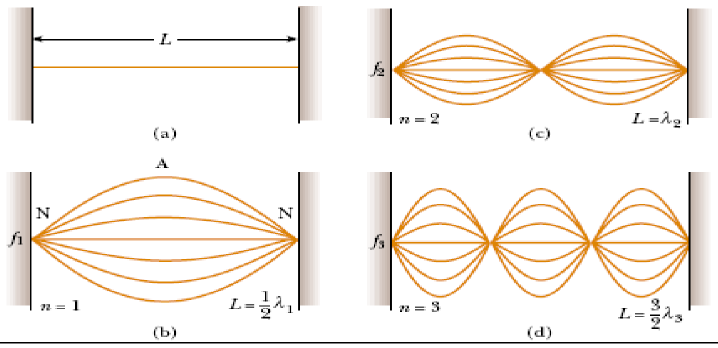


Figure 4.7

(a) A string of length L fixed at both ends.
(b) The fundamental, or the first harmonic
(c) The second harmonic
(d) The third harmonic.

The first normal mode Fig (4.7 b), has nodes at its ends and one antinode in the middle. This is the longest wavelength mode. The first normal mode occurs when the length of the string is half the wavelength λ_1 or $\lambda_1 = 2L$

The second normal mode Fig (4.7 c) occurs when the wavelength equals the length of the string, $\lambda_2 = L$

The third normal mode Fig (4.7 d) occurs when the wavelength equals 2L/3.

In general, the wavelengths of the various normal modes for the string of length L fixed at both ends

$$\lambda_n = \frac{2L}{n} \qquad n = 1,2,3,\ldots\ldots\ldots \qquad (4\text{-}3)$$

The natural frequencies associated with these modes are obtained from the relationship $v = \lambda f$, using equation (4.3), we find the natural frequencies are

$$f_n = \frac{v}{\lambda_n} = n\left(\frac{v}{2L}\right) \qquad n = 1,2,3,\ldots\ldots \qquad (4\text{-}4)$$

Because: $v = \sqrt{\frac{T}{\mu}}$ **Where** T is the tension in the string

μ is the linear mass density

So $$f_n = \frac{n}{2L}\sqrt{\left(\frac{T}{\mu}\right)} \qquad n = 1,2,3,\ldots\ldots \qquad (4\text{-}5)$$

The lowest frequency f1, which corresponds to n =1 is called the fundamental frequency and given by $f_1 = \frac{1}{2L}\sqrt{\left(\frac{T}{\mu}\right)} \qquad n = 1$

Example 4.2:

Middle C on a piano has a fundamental frequency of 262 HZ, and the first A above middle C has a fundamental frequency of 440 Hz.

(a) Calculate the frequencies of the next two harmonics of the C string.

(b) If the A and C strings have the same linear mass density μ and length L,

(c) determine the ratio of tensions in the two strings.

Solution(a):

We know that the frequencies of higher harmonics are integer multiples of the fundamental frequency $f1$ so we find that

$$f_2 = 2f_1 = 524 \text{ HZ}$$

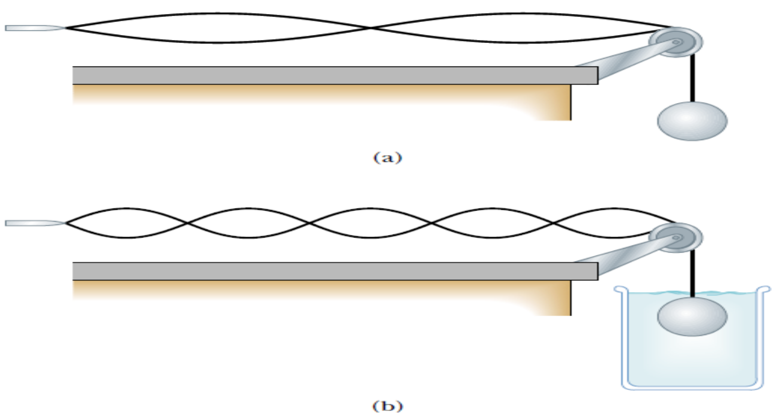
$$f_3 = 3f_1 = 786 \text{ HZ}$$

Solution (b): $f_{1A} = \frac{1}{2L}\sqrt{\left(\frac{T_A}{\mu}\right)}$ *and* $f_{1C} = \frac{1}{2L}\sqrt{\left(\frac{T_C}{\mu}\right)}$

$$\frac{f_{1A}}{f_{1C}} = \sqrt{\frac{T_A}{T_C}} \rightarrow \frac{T_A}{T_C} = \left(\frac{f_{1A}}{f_{1C}}\right)^2 = 2.82$$

Example 4.3:

One end of a horizontal string is attached to a vibrating blade and the other end passes over a pulley as in the following figure. A sphere of mass 2 Kg hangs on the end of the string. The string is vibrating in its second harmonic. A container of water is raised under the sphere so that the sphere is completely submerged. After this is done, the string vibrates in its fifth harmonic. What is the radius of the sphere?



Solution:

When the sphere is immersed in the water. The buoyant force acts upward on the sphere, reducing the tension in the string. The change in tension causes a change in the speed of waves on the string, which in turn causes a change in the wavelength this altered wavelength results in the string vibrating in its fifth normal mode rather than the second. Apply Newton's second law

$\sum F = T_1 - mg = o$

$T_1 = mg = 2*9.8 = 19.6$

As soon as the sphere is immersed in water, the tension in the string decreases to T2, by applying Newton's second law to the sphere we have

$T2 + B - mg = 0 \longrightarrow B = T2 - mg$

The frequency of a standing wave on a string before we immerse the sphere and after we immerse are,

$$f = \left(\frac{n_1}{2L}\right)\sqrt{\left(\frac{T_1}{\mu}\right)} \quad and \quad f = \left(\frac{n_2}{2L}\right)\sqrt{\left(\frac{T_2}{\mu}\right)}$$

Dividing the last two equations we get $1 = \frac{n_1}{n_2}\sqrt{\frac{T_1}{T_2}}$

Where the frequency is the same in both cases, because it is determined by the vibrating blade, and the linear mass density μ and the Length L of the vibrating portion of the spring are the same in both cases.

$$T_2 = \left(\frac{n_1}{n_2}\right)^2 T_1 = 3.14N \qquad So \quad B = mg - T_2 = 16.5N$$

Expressing the buoyant force in terms of the radius of the sphere

$$B = \rho_{water} g V_{spher} = \rho_{water} g (4\pi r^3 / 3)$$

$$r = 7.38cm$$

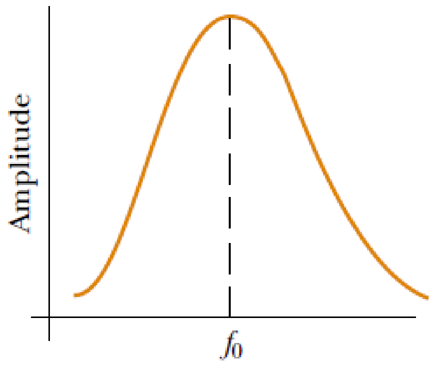
4.5 Resonance:

We have seen that a system such as a taut string is capable of oscillating in one or more normal modes

Resonance phenomenon:

If a periodic force is applied to a system, the amplitude of the resulting motion is greatest when the frequency of the applied force is equal to one of the natural frequencies of the system, and this frequency is called resonance frequency.

The opposite figure shows the response of an oscillating system to various frequencies, where one of the resonance frequencies is denoted by $f_\circ$.



4.6 standing Waves in Air Columns:

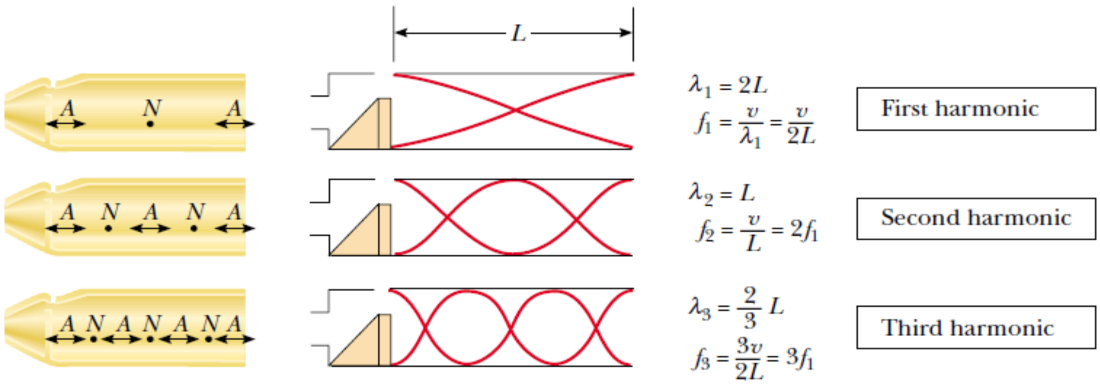
Standing waves can be setup in a tube of air, as the result of interference between two longitudinal sound waves traveling in opposite directions.

In a pipe closed at one end, the closed end is a displacement node because the wall at this end doesn't allow longitudinal motion of the air. So at a closed end of a pipe, the reflected sound wave is 180° out of phase with the incident wave.

The open end of an air column is approximately a displacement antinode. With the boundary conditions of nodes or antinodes at the ends of the air column, we have a set of normal modes of oscillation.

The first three normal modes of oscillation of a pipe open at both ends are shown in figure 4.9a. In the first normal mode, the standing wave extends between two adjacent antinodes, which is a distance of half a wavelength. So the wavelength is twice the length of the pipe, and the fundamental frequency is f_1=v/2L

Figure 4.8a Opened pipe at both ends



(a) Open at both ends

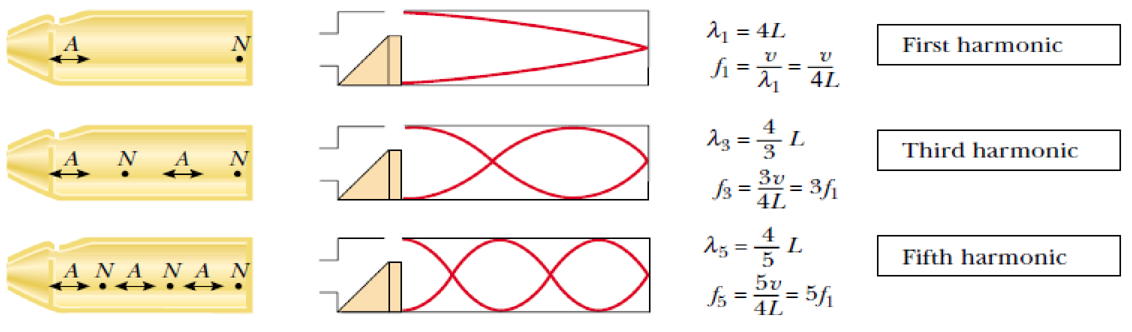
We can express the natural frequencies of oscillation as

$$f_n = {}^{nv}\!/_{2L}$$

If a pipe is closed at one end and open at the other, the closed end is a displacement node. The standing wave for the fundamental mode extends from an antinode to the adjacent node, which is one fourth of a wavelength. So the wavelength for the first normal mode is 4L, and the fundamental frequency is

$$f_1 = {}^{v}\!/_{4L}$$

Figure 4.8b Aclosed pipe at one end and open at the other



(b) Closed at one end, open at the other

Example 4.5:
A section of drainage 1.23m in length makes a howling noise when the wind blows.

(a) Determine the frequencies of the first three harmonics of the culvert if it is cylindrical in shape and open at both ends. Take v= 343 m/s as the speed of sound in air.
(b) What are the three lowest natural frequencies of the culvert if it is blocked at one end ?
(c) For the culvert open at both ends, how many of the harmonics present fall within the normal human hearing range (20 to 20000 HZ)

Solution(a):
The frequency of the first harmonic of a pipe open at both ends is

f_1= v/2L

= 343/ (2* 1.23) = 139 HZ

Because both ends are open, all harmonics are present; thus

f_2 = 2 f_1= 278 HZ & f_3 = 3f_1= 417 HZ

Solution(b):
The fundamental frequency of a pipe closed at one end is

f_1= v/4L= 343/ (4*1.23) = 69.7 HZ

In this case, only odd harmonics are present; the next two harmonics have frequencies

f_3 = 3f_1= 209 HZ & f_5 =5f_1= 349 HZ

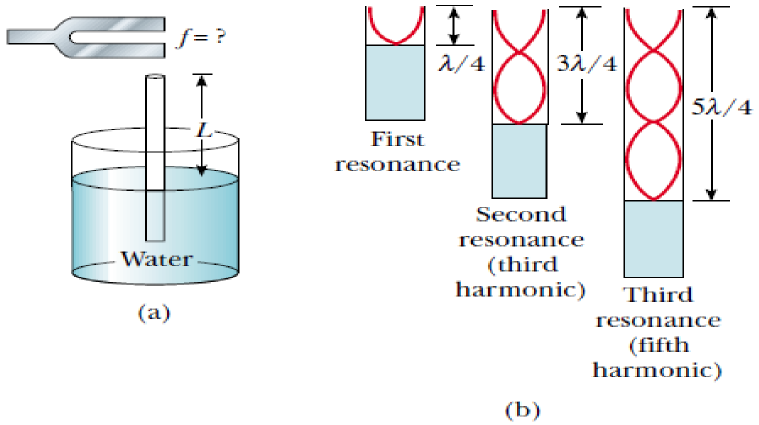
Solution(c):
Because all harmonics are present for a pipe open at both ends, we can express the frequency of the highest harmonic heard as $f_n = n f 1$ where n is the number of harmonics that we can hear. For f_n= 20000Hz we find that the number of harmonics present in the audible range is

$$n= (20000)/ (139) = 143$$

So the first few harmonics are be heared.

Example 4.6:
A simple apparatus for demonstrating resonance in an air column is depicted in the following figure. A vertical pipe open at both ends is partially submerged in water, and a tuning fork vibrating at an unknown frequency is placed near the top of the pipe. The length L of the air column can be adjusted by moving the pipe vertically. The sound waves generated by the fork are reinforced when L corresponds to one of the resonance frequencies of the pipe.

Figure 4.9
(a) apparatus fordemonstrating the resonance of sound waves in a pipe closed at one end.
(b) The first three normal modes of the system in part a



For a certain pipe, the smallest value of L for which a peak occurs in the sound intensity is 9cm what are

(a) the frequency of the tuning fork
(b) the value of L for the next two resonance frequencies?

Solution:
(a) although the pipe is open at its lower end to allow the water to enter, the water's surface acts like a wall at one end(i.e closed air column at one end) so the fundamental frequency is given by f_1= v/4L

$$\text{So } f_1 = 343/4(0.09) = 953 \text{ HZ}$$

(b) we know from figure 18.9b the wavelength of the fundamental mode is

$$\lambda = 4L = 4(0.09) = 0.36\text{m}$$

the next two normal modes corresponds to lengths of
$L= 3\lambda/4 = 0.270$m & $L= 5\lambda/4 = 0.450$m.

Summary:

1- When two traveling waves having equal amplitudes and frequencies superimpose , the resultant waves has an amplitude that depends on the phase angle $\emptyset$ between the resultant wave has an amplitude that depends on the two waves are in phase , two waves . **Constructive interference** occurs when the two waves are in phase, corresponding to $\emptyset = 0.2\pi, 4\pi, \ldots$ rad **Destructive interference** occurs when the two waves are 180^o out of phase, corresponding to $\emptyset = \pi.3\pi, 5\pi, \ldots$ rad.

2- Standing waves are formed from the superposition of two sinusoidal waves having the same frequency , amplitude , and wavelength but traveling in opposite directions . the resultant standing wave is described by $y = (2A sin\, kx) \cos \omega t$

Hence the amplitude of the standing wave is 2A, and the amplitude of the simple harmonic motion of any particle of the medium varies according to its position as 2A sin kx.The points of zero amplitude(called **nodes**)occur at $x = n\lambda/2 (n = 0,1,2,3, \ldots)$ the maximum amplitude points (called **antinodes**) at $x = n\lambda/4 (n = 0,1,3,,5 \ldots)$ .Adjacent antinode are separated by a distance $\boldsymbol{\lambda/2}$ Adjacent nodes also are separated by a distance $\boldsymbol{\lambda/2}$**.**

3- The natural frequencies of vibration of a string of length L and fixed at both ends are quantized and are given by

$$f_n = \frac{n}{2l}\sqrt{\frac{T}{\mu}} \qquad n = 1,2,3, \ldots \ldots$$

Where T is the tension in the string and μ is its linear mass density .The natural frequencies of vibration $f_1{}^2 f_1{}^3 f_1, \ldots..$ form a **harmonic series**.

4- Standing waves can be produces in a column of air inside a pipe. If the pipe is open at both ends, all harmonics are present and the natural frequencies of oscillation are $fn = n\frac{v}{2L} \qquad n = 1,2,3, \ldots$

If the pipe is open at one end and closed at the other, only the odd harmonics are present, and the natural frequencies of oscillation are

$$f\mathrm{N} = n\frac{v}{4L} \qquad n = 1,3,5, \ldots$$

5- An oscillating system is in **resonance** with some driving force whenever the frequency of the driving force matches one of the natural frequencies of the system. When the system is resonating, it responds by oscillating with a relatively large amplitude.