بسم الله الرحمن الرحيم

Vibration

Fayoum University

Faculty of Engineering
Mechanical Engineering Dept.

Lecture (3)

on

The Vibrations of Systems Having Single Degree of Freedom- Response to Initial Excitation

By

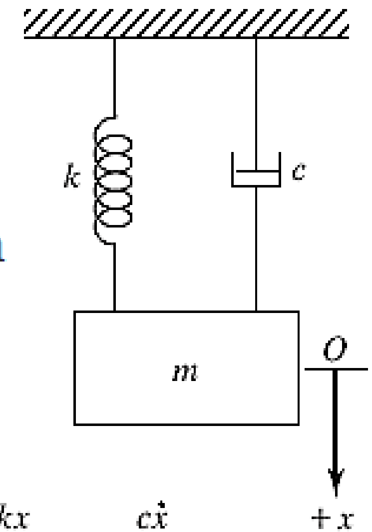
Dr. Emad M. Saad

Mechanical Engineering Dept.
Faculty of Engineering
Fayoum University

2015 - 2016

Free Vibration with Viscous Damping

$$\pm\sum F_x = m\,a_x\,; \qquad -k\,x - c\dot{x} = m\,\ddot{x}$$



$$m\ddot{x} + c\dot{x} + k\,x = 0$$

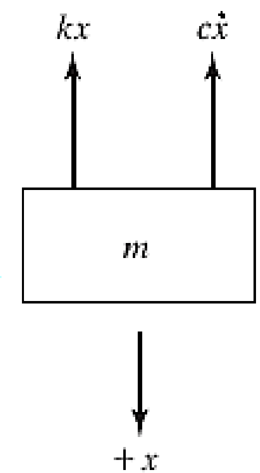
This linear, second-order, homogeneous, differential equation has a solution of the form

$$x = e^{\lambda t}$$

e is the base of the natural logarithm and λ is a constant.

$$m\,\lambda^2\,e^{\lambda t} + c\,\lambda\,e^{\lambda t} + k\,e^{\lambda t} = 0$$

$$e^{\lambda t}\left(m\,\lambda^2 + c\,\lambda + k\right) = 0$$



Since $e^{\lambda t}$ can never be zero, a solution is possible provided

$$m\,\lambda^2 + c\,\lambda + k = 0$$

Free Vibration with Viscous Damping

$$\lambda_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \qquad \lambda_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

These roots give two solutions

$x_1 = e^{\lambda_1 t}$ and $x_2 = e^{\lambda_2 t}$ So, the general solution

$$x = A\,e^{\lambda_1 t} + B\,e^{\lambda_2 t}$$

where A and B are arbitrary constants to be determined from the initial conditions of the system.

There are three possible combinations of λ_1 and λ_2 which must be considered. Before discussing these combinations, however, we will first define the critical damping coefficient c_{crit} and the damping ratio.

Free Vibration with Viscous Damping

The critical damping is defined as the value of c which makes the radical in the last equations equal to zero; i.e.,

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} = 0 \quad \text{or} \quad c_{crit} = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

For any damped system, the damping ratio ζ is defined as the ratio of the damping constant to the critical damping constant:

$$\zeta = \frac{c}{c_{crit}} = \frac{\text{Damping constant}}{\text{Damping constant for critically damped condition}}$$

$$\frac{c}{2m} = \frac{c}{c_{crit}} \cdot \frac{c_{crit}}{2m} = \zeta\omega_n$$

$$\lambda_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

$$x = A\,e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + B\,e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

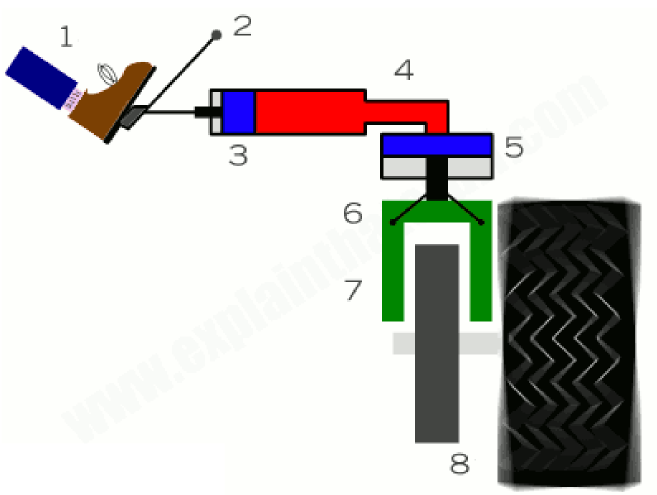
Free Vibration with Viscous Damping

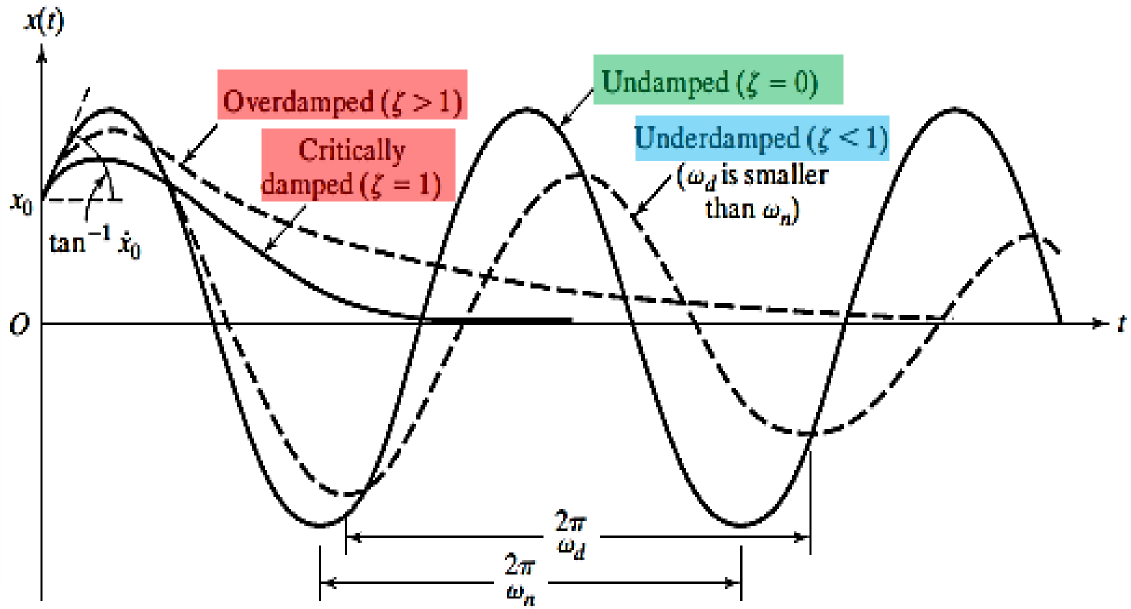
The nature of the roots λ_1 and λ_2, hence the behavior of the solution depends upon the magnitude of damping.

$$\zeta = \frac{c}{c_{crit}}$$

When $\zeta = 0$ leads to the undamped vibrations

When $\zeta \neq 0$ and consider the following three cases.





Free Vibration with Viscous Damping

Case 1: Overdamped system

When $c > c_{\text{crit}}$ the roots λ_1 and λ_2 are both real. The general solution of Eq. (2.21) can then be written as

$$x = A\,e^{\lambda_1 t} + B\,e^{\lambda_2 t} = 0$$

Motion corresponding to this solution is non-vibrating. The effect of damping is so strong that when the block is displaced and released, it simply creeps back to its original position without oscillating. The system is said to be overdamped as indicated in Figure (2.6).

Free Vibration with Viscous Damping

Case 2: Critically damped system

If $c = c_{\text{crit}}$ then $\lambda_1 = \lambda_2 = c_{crit}/2m = -\omega_n$ This situation is known as critical damping, since it represents a condition where c has the smallest value necessary to cause the system to be non-vibrating. Using the methods of differential equations, it can be shown that the solution to Eq. (2.21) for critical damping is

$$x = (A + Bt)e^{-\omega_n t}$$

Free Vibration with Viscous Damping

Case 3: Underdamped system

Most often $c < c_{\text{crit}}$ in which case the system is referred to as underdamped. In this case the roots λ_1 and λ_2 are complex numbers, and it can be shown that the general solution of Eq. (2.21) can be written as

$$x = D\left[e^{-(c/2m)t}\sin(\omega_d t + \phi)\right]$$

where D and ϕ are constants generally determined from the initial conditions of the problem. The constant ω_d is called the damped natural frequency of the system. It has a value of

$$\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = \omega_n\sqrt{1 - \left(\frac{c}{c_{crit}}\right)^2} = \omega_n\sqrt{1-\zeta^2}$$

Note: When $\zeta = 1$, one has the critically damped response because below this value, the response is oscillatory (underdamped), and above this value, the response is nonoscillatory (overdamped).

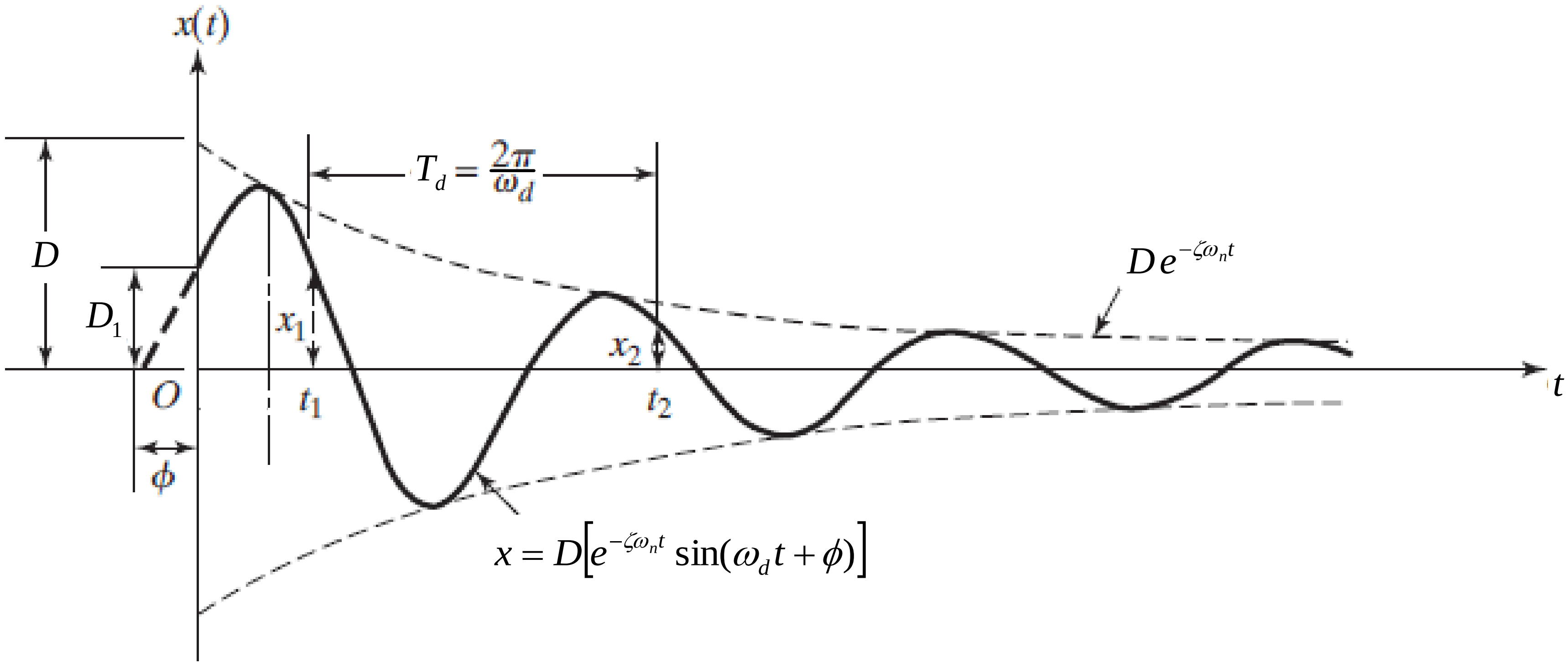
Free Vibration with Viscous Damping

The graph of Eq. (2.29) is shown in Figure (2.7). The initial limit of motion, D, diminishes with each cycle of vibration, since motion is confined within the bounds of the exponential curve. Using the damped natural frequency ω_d, the period of damped vibration can be written as

$$T_d = \frac{2\pi}{\omega_d}$$

Since $\omega_d < \omega_n$, Eq. (2.30), the period of damped vibration, T_d , will be greater than that of free vibration, $T = 2\pi/\omega_n$.

Free Vibration with Viscous Damping



Free Vibration with Viscous Damping

The variation of ω_d/ω_n, with ζ is shown in Figure (2.8). In addition, Figure (2.9) shows the time variation of displacement as a function of damping ratio $\zeta < 1$ (underdamping).

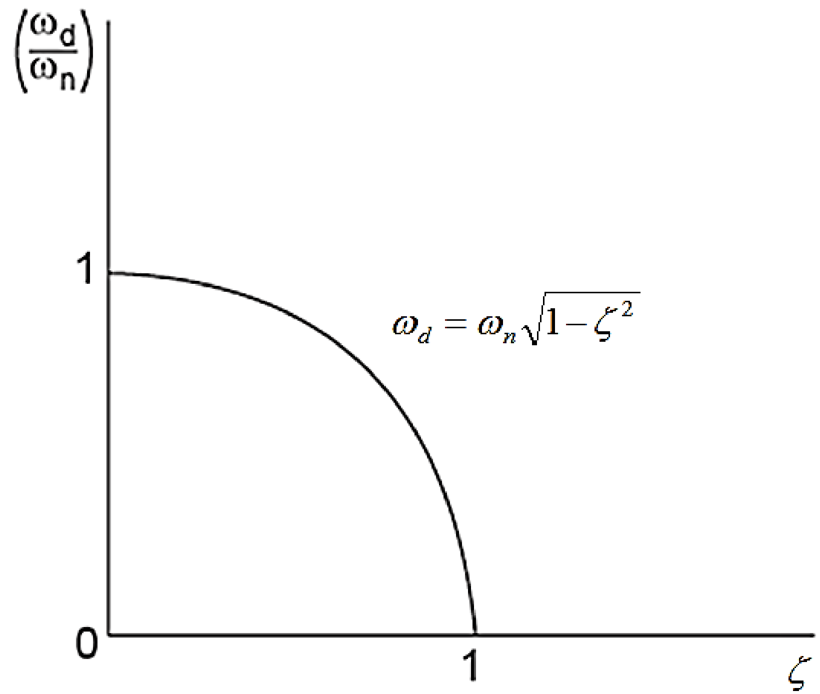


Fig.2.8: variation of ω_d/ω_n, with damping factor ζ .

Free Vibration with Viscous Damping

Logarithmic decrement

$$\delta = \ln(x_n / x_{n+1})$$

From Eq. (2.29), x is given by

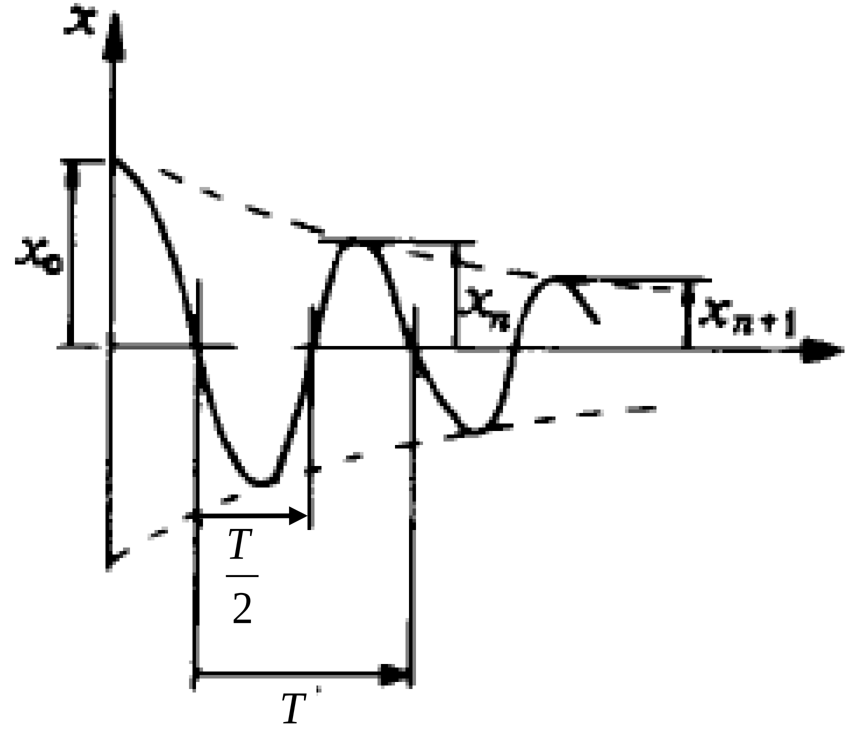
$$x = e^{-\zeta\omega_n t} f(\omega t) = e^{-\zeta\omega_n t} \sin\left[\left(\omega^2 - \frac{c^2}{4m^2}\right)^{1/2} + \phi\right]$$

But

$$\frac{x_n}{x_{n+1}} = \frac{e^{-\zeta\omega_n t} f(\omega t)}{e^{-\zeta\omega_n (t+T)} f(\omega t + T)} = e^{\zeta\omega_n T}$$

Thus

$$\delta = \zeta\omega_n T = \frac{\zeta\omega_n 2\pi}{\left[\omega_n\sqrt{(1-\zeta^2)}\right]} = \frac{2\pi\zeta}{\sqrt{(1-\zeta^2)}}$$



where T is the period of damped oscillation.

If the amount of damping present is small compared to the critical damping, T approximates to $T = 2\pi/\omega_n$, and then

$$\delta \approx 2\pi\zeta = \frac{c\pi}{m\omega_n}$$

Free Vibration with Viscous Damping

Specific loss

A further way of indicating the amount of damping in lightly damped systems is to evaluate the energy lost per cyc1e as a fraction of the energy at the start of the cycle.

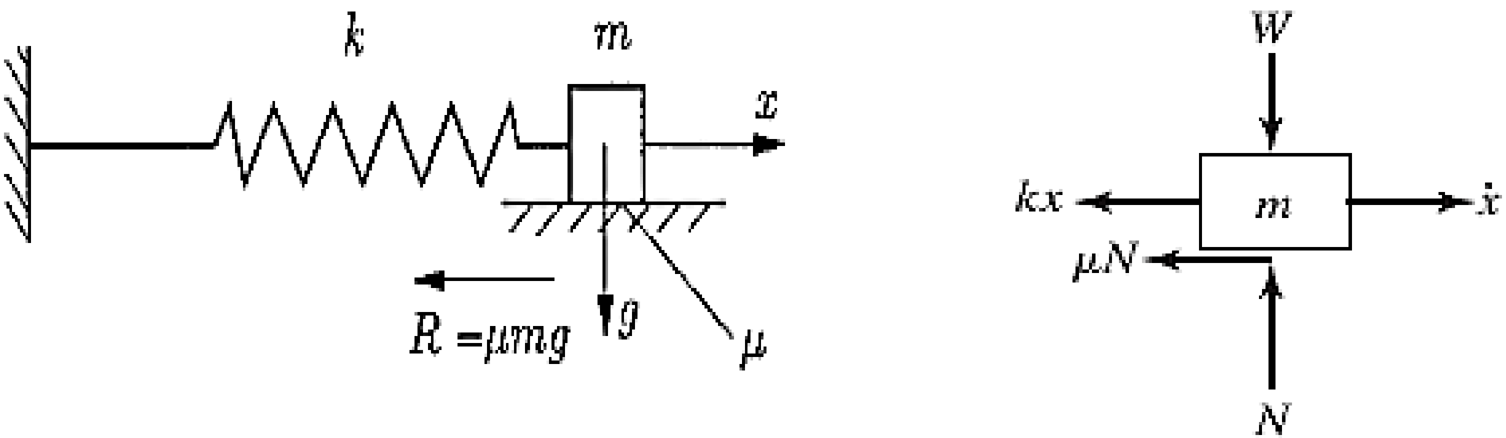
$$Specific\ loss = \frac{\frac{1}{2}kx_n^2 - \frac{1}{2}kx_{n+1}^2}{\frac{1}{2}kx_n^2} = 1 - \left(\frac{x_{n+1}}{x_n}\right)^2 = 1 - \exp[-2\zeta\omega_n\tau]$$

So, for small damping,

$$Specific\ loss = 1 - \exp[-2\zeta\omega_n\tau + \cdots\cdots] \approx 2\zeta\omega_n\tau \approx 4\pi\zeta \approx 2\delta$$

Free Vibration with Viscous Damping

Damped Free Vibration with Dry Damping (Coulomb's damping)



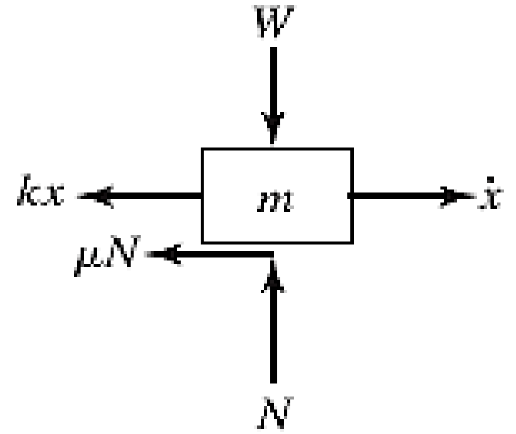
N is the normal force, equal to the weight of the mass (***W = mg***) coefficient of friction (***μ***) depends on the materials in contact and the condition of the surfaces in contact.

For example, ***μ*** ≈ 0.1 for metal on metal (lubricated), 0.3 for metal on metal (unlubricated), and nearly 1.0 for rubber on metal.

Free Vibration with Viscous Damping

Damped Free Vibration with Dry Damping (Coulomb's damping)

the viscous damping force R is given by Coulomb's law of friction.



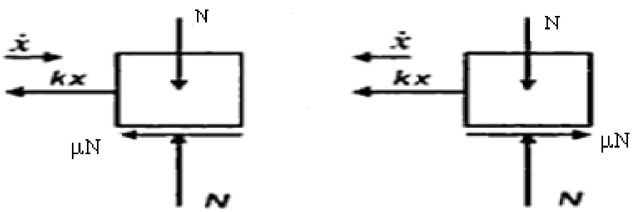
$$R = \mu F_n$$

F_n is the normal force and μ is the coefficient of friction.

$$m\ddot{x} + kx = \mu mg$$

The general solutions of the above equation are

$$\begin{cases} x = A\sin\omega_n t + B\cos\omega_n t - \dfrac{\mu mg}{k} \\ x = A\sin\omega_n t + B\cos\omega_n t + \dfrac{\mu mg}{k} \end{cases}$$



where the natural angular frequency is $\omega_n = \sqrt{k/m}$.

The sign of last term in the above equation is depending on the direction of motion of the block as shown in the figure

Free Vibration with Viscous Damping

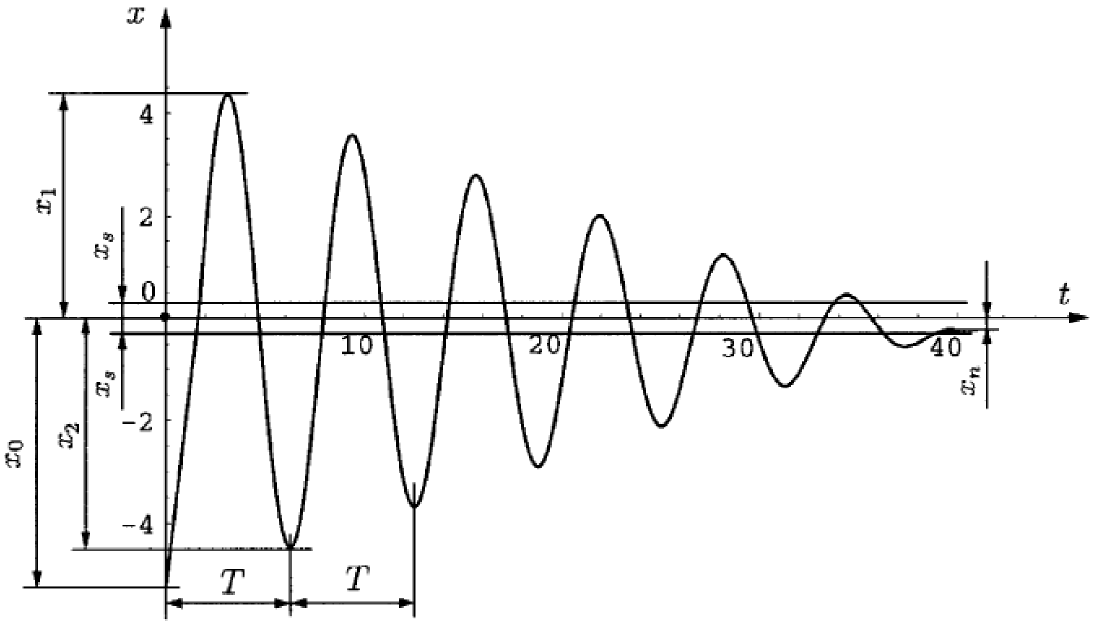
Damped Free Vibration with Dry Damping (Coulomb's damping)

Assume the body of mass m at rest and the spring is compressed (or stretched) such that its initial displacement is $|x_0| > |x_s = \mu mg/k|$. With the initial conditions

$$t = 0 \Rightarrow \begin{cases} x = \pm x_0 \\ \dot{x} = 0 \end{cases}$$

The equations of motion are

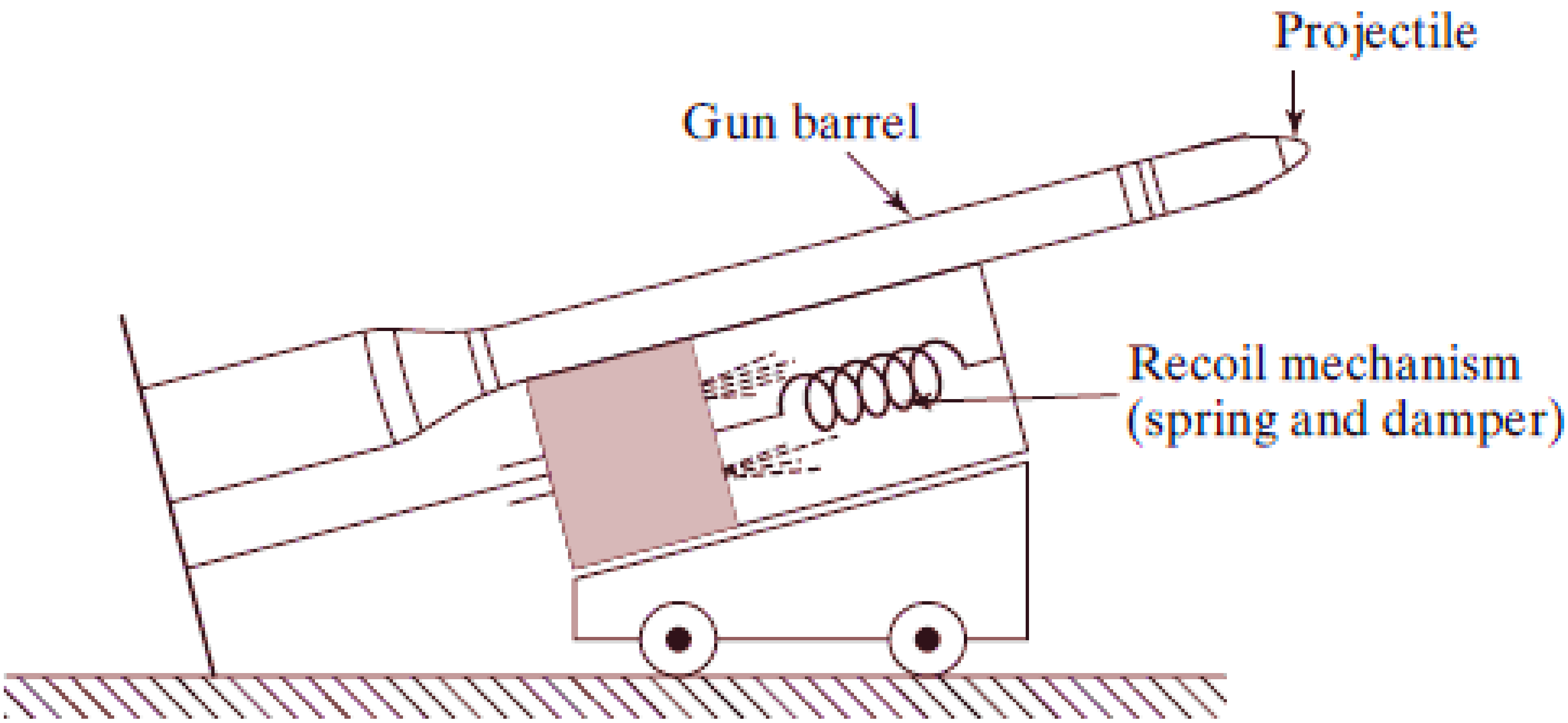
$$\begin{cases} x = -(x_0 - x_s)\cos\omega_n t - x_s \\ x = (x_0 - 3x_s)\cos\omega_n t + x_s \end{cases}$$



Solved Example:1 (Free Vibration with Viscous Damping)

The schematic diagram of a large cannon is shown in Figure (E2.6). When the gun is fired, high pressure gases accelerate the projectile inside the barrel to a very high velocity. The reaction force pushes the gun barrel in the direction opposite that of the projectile. Since it is desirable to bring the gun barrel to rest in the shortest time without oscillation, it is made to translate backward against a critically damped spring-damper system called the *recoil mechanism*. In a particular case, the gun barrel and the recoil mechanism have a mass of 500 kg with a recoil spring of stiffness 10,000 N/m. The gun recoils 0.4 m upon firing. Find (a) the critical damping coefficient of the damper, (b) the initial recoil velocity of the gun, and (c) the time taken by the gun to return to a position 0.1 m from its initial position.

Solved Example:1 (Free Vibration with Viscous Damping)



Solved Example:1 (Free Vibration with Viscous Damping)

The undamped natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10000}{500}} = 4.4721 \; rad/s$$

and the critical damping coefficient of the damper is

$$c = 2m\omega_n = 2\,(500)(4.4721) = 4472.1 \; N.s/m \qquad \textbf{Ans. (a)}$$

The response of a critically damped system is given by Eq. (2.28):

$$x = (A + Bt)e^{-\omega_n t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (E.1)$$

where $A = x_0$ and $B = \dot{x}_0 + \omega_n x_0$. The time t_1 at which $x(t)$ reaches a maximum value can be obtained by setting $\dot{x}(t) = 0$. The differentiation of Eq. (E.1) gives

$$\dot{x}(t) = B\,e^{-\omega_n t} - \omega_n (A + Bt)e^{-\omega_n t}$$

Solved Example:1 (Free Vibration with Viscous Damping)

Hence $\dot{x}(t)=0$ yields

$$t_1=\frac{1}{\omega_n}-\frac{A}{B} \quad \text{(E.2)}$$

In this case $x_0=A=0$, hence Eq. (E.2) leads to $t_1=\frac{1}{\omega_n}$. Since the maximum value of $x(t)$ or the recoil distance is given to be $x_{max}=0.4$ m, we have

$$x_{max}=x(t=t_1)=Bt_1\,e^{-\omega_n t_1}=\frac{\dot{x}_0}{\omega_n}e^{-1}=\frac{\dot{x}_0}{e\,\omega_n}$$

or

$$\dot{x}_0=x_{max}\,\omega_n e=(0.4)(4.4721)(2.7183)=4.8626\ m/s \qquad \textit{Ans. (b)}$$

If t_2 denotes the time taken by the gun to return to a position 0.1 m from its initial position, we have

$$0.1=Bt_2\,e^{-\omega_n t_2}=4.8626\,t_2\,e^{-4.4721\,t_2} \quad \text{(E.3)}$$

The solution of Eq. (E.3) gives t_2=0.8258 s. *Ans. (c)*


QUIZ

Quiz

A railroad car of mass 2,000 kg traveling at a velocity v = 10 m/s is stopped at the end of the tracks by a spring-damper system, as shown in Figure (P2.13). If the stiffness of the spring is k = 80 N/mm and the damping constant is c = 20 N.s/m determine (a) the maximum displacement of the car after engaging the springs and damper and (b) the time taken to reach the maximum displacement.

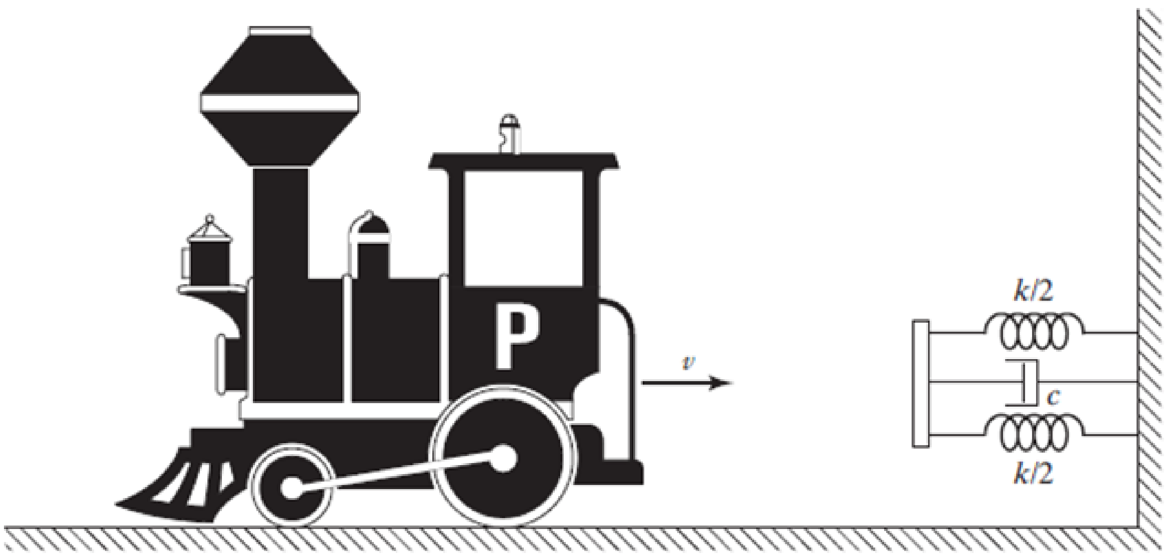


Fig. P2.13

Thank you