بسم الله الرحمن الرحيم

Vibration

Fayoum University

Faculty of Engineering
Mechanical Engineering Dept.

Lecture (4)

on

The Vibrations of Systems Having Single Degree of Freedom- Response to Harmonic and Periodic Excitations

By

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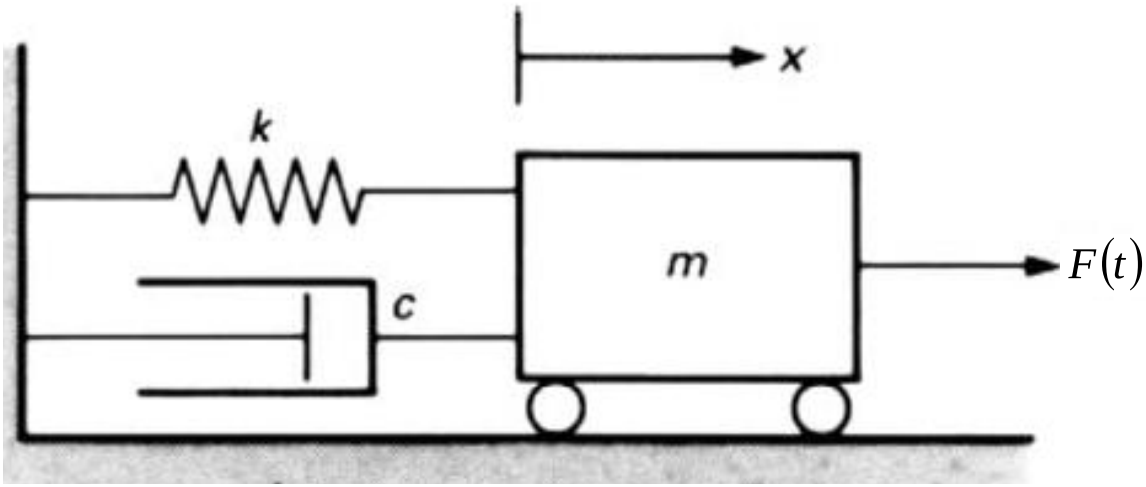
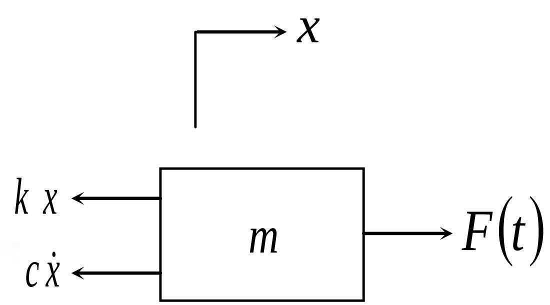
Damped Systems due to a Simple Harmonic Exciting Force with Constant Amplitude





$$\pm\sum F_x = m\,a_x\,;$$

$$F(t) - k\,x - c\dot{x} = m\,\ddot{x}$$

$$\ddot{x} + \frac{k}{m}x + \frac{c}{m}\dot{x} = \frac{F(t)}{m}$$

Damped Systems due to a Simple Harmonic Exciting Force with Constant Amplitude

$$\ddot{x}+\frac{k}{m}x+\frac{c}{m}\dot{x}=\frac{F(t)}{m}$$

If $F(t)$ is sinusoidal

$m\ddot{x}+c\dot{x}+kx=F_0\sin\omega_0 t$ Where ω_0 is excitation frequency

The sustained motion is given by the particular solution.

$$x=X\sin(\omega_0 t-\phi)$$

$$\dot{x}=X\omega_0\cos(\omega_0 t-\phi)=X\omega_0\sin\left(\omega_0 t-\phi+\frac{1}{2}\pi\right)$$

and

$$\ddot{x}=-X\omega_0^2\sin(\omega_0 t-\phi)=X\omega_0^2\sin(\omega_0 t-\phi+\pi)$$

then

$$Xm\,\omega_0^2\sin(\omega_0 t-\phi+\pi)+X\,c\omega_0\sin(\omega_0 t-\phi+\pi/2)+X\,k\sin(\omega_0 t-\phi)=F_0\sin\omega_0 t$$

Damped Systems due to a Simple Harmonic Exciting Force with Constant Amplitude

If $F(t)$ is sinusoidal

$$Xm\,\omega_0^2 \sin(\omega_0 t - \phi + \pi) + X\,c\omega_0 \sin(\omega_0 t - \phi + \pi/2) + X\,k \sin(\omega_0 t - \phi) = F_0 \sin \omega_0 t$$

From the diagram,

$$F_0^2 = (kX - mX\omega_0^2)^2 + c^2 X^2 \omega_0^2$$

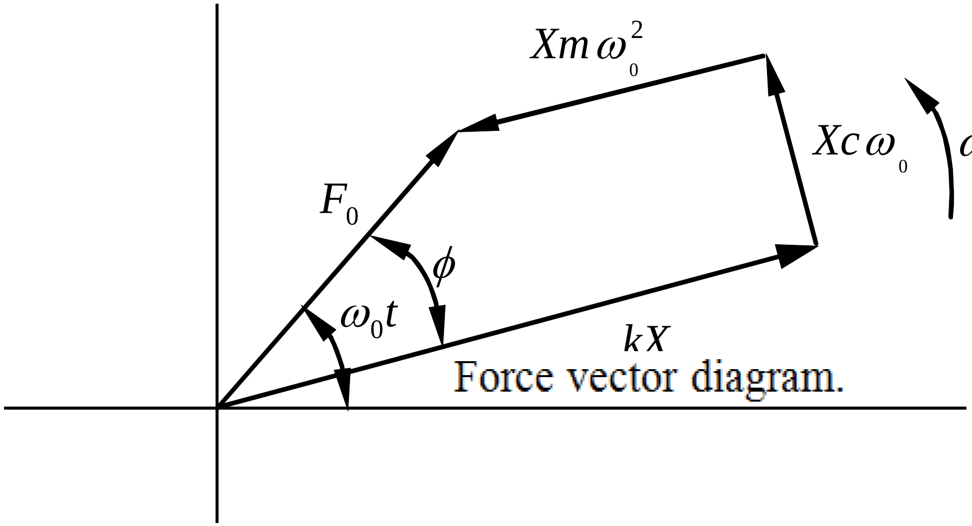
or

$$X = \frac{F_0}{\sqrt{(k - m\omega_0^2)^2 + c^2\omega_0^2}}$$

and

$$\tan\phi = \left[\frac{cX\omega_0}{kX - mX\omega_0^2}\right]$$

$$\phi = \tan^{-1}\left[\frac{c\omega_0}{k - m\omega_0^2}\right]$$



Force vector diagram.

Damped Systems due to a Simple Harmonic Exciting Force with Constant Amplitude

If $F(t)$ is sinusoidal

Since $\omega_n = \sqrt{k/m}$ and $c_{crit} = 2m\omega_n$, then the above equations can also be written as

$$X = \frac{F_0/k}{\sqrt{\left[1-(\omega_0/\omega_n)^2\right]^2 + \left[2(c/c_{crit})(\omega_0/\omega_n)\right]^2}}$$

$$\phi = \tan^{-1}\left[\frac{\left[2(c/c_{crit})(\omega_0/\omega_n)\right]}{1-(\omega_0/\omega_n)^2}\right]$$

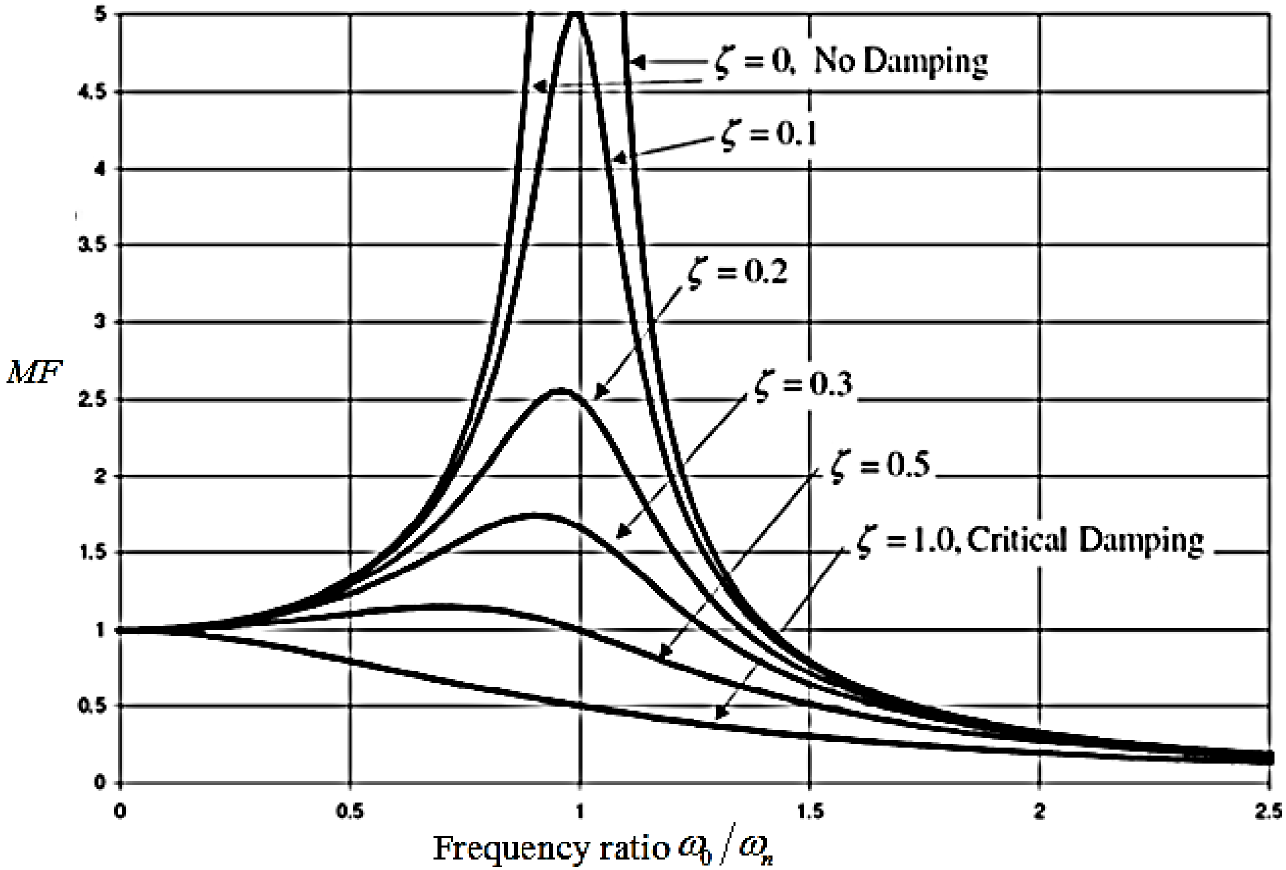
Thus the steady-state solution is

$$x(t) = \frac{F_0/k}{\sqrt{\left[1-(\omega_0/\omega_n)^2\right]^2 + \left[2(c/c_{crit})(\omega_0/\omega_n)\right]^2}}\sin(\omega_0 t - \phi)$$

The angle ϕ represents the phase difference between the applied force and the resulting steady-state vibration of the damped system.

Damped Systems due to a Simple Harmonic Exciting Force with Constant Amplitude

If $F(t)$ is sinusoidal

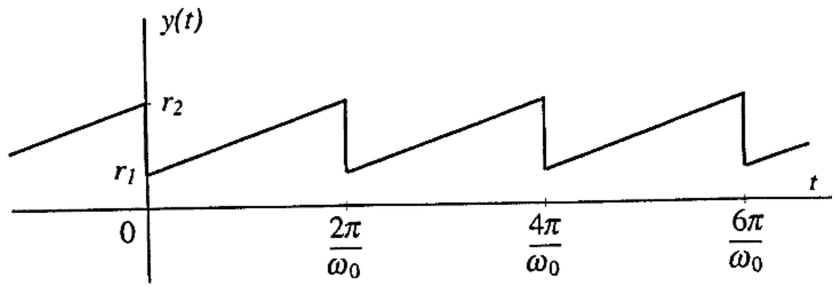


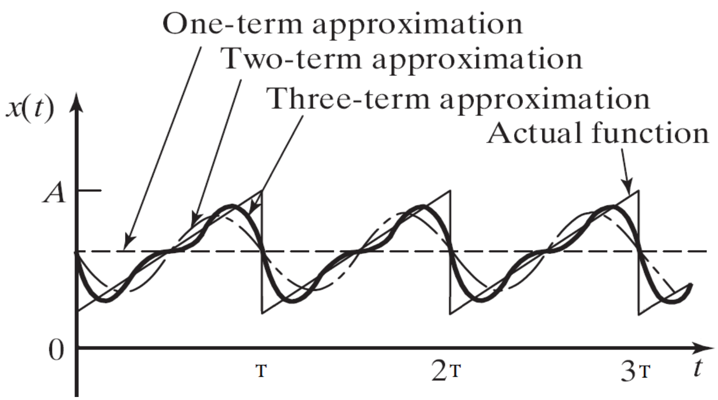
Damped Systems due to a Simple Harmonic Exciting Force with Constant Amplitude

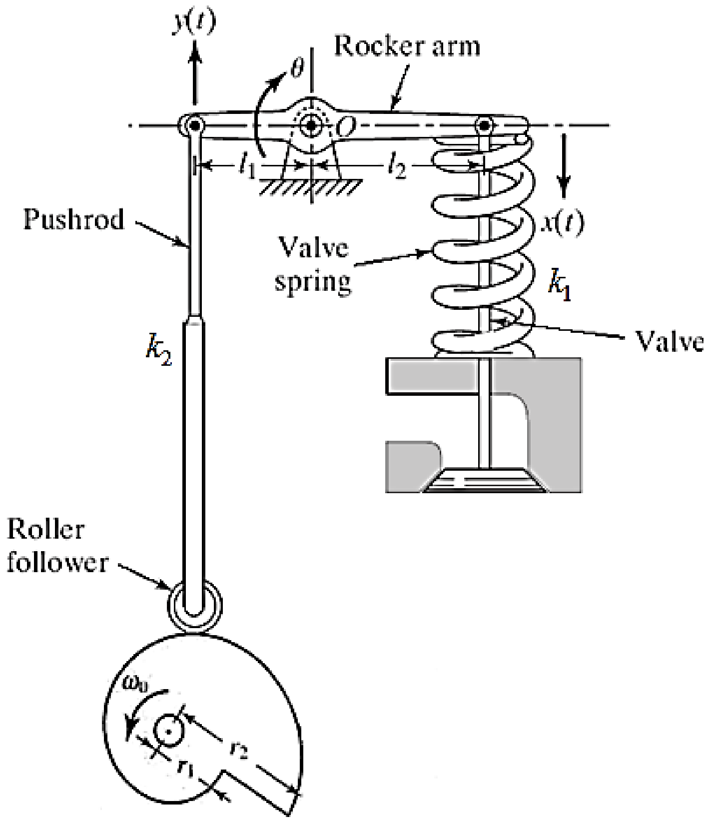
If $F(t)$ is non-sinusoidal

It is not difficult to verify that the equation of motion for the system is

$$m\ddot{x} + c\dot{x} + (k_1 + k_2)x = k_2 y$$







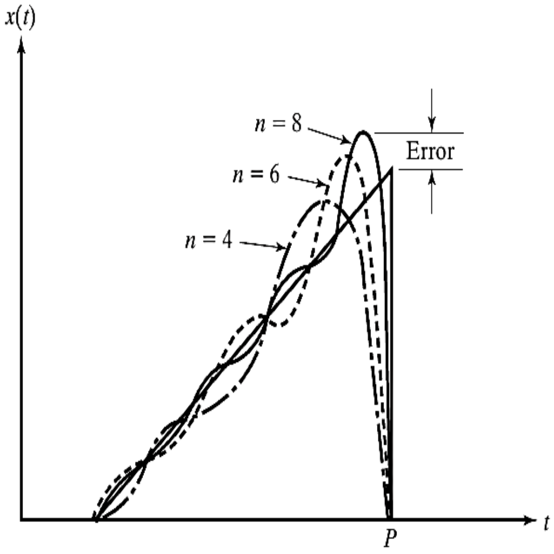
Damped Systems due to a Simple Harmonic Exciting Force with Constant Amplitude

If $F(t)$ is non-sinusoidal

The function that displayed in last figure, can be expanded in the trigonometric form of Fourier series

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right), \quad \omega_0 = 2\pi/T$$

Where n =1, 2, …… is an integer number of terms of Fourier series and ω_0 is called the fundamental frequency.



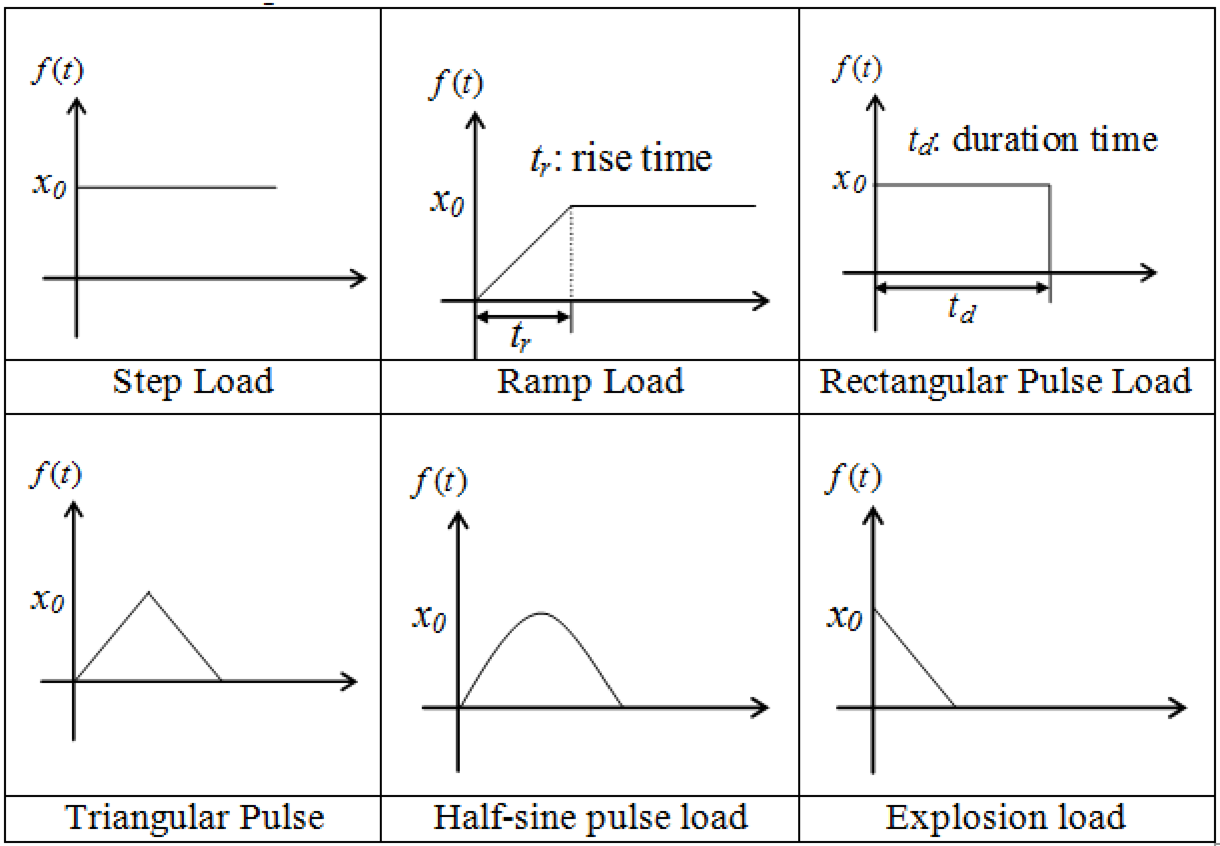
The coefficients a_n (n=1, 2, ……) and b_n (n=1, 2, ……) are known as Fourier coefficients and, provided $F(t)$ is specified as a function of time over a full period, that can be calculated by means of the formulas

$$a_n = \frac{2}{T}\int_0^T F(t)\cos n\omega_0 t\, dt, \quad n = 0, 1, \ldots\ldots\ldots\ldots$$

$$b_n = \frac{2}{T}\int_0^T F(t)\sin n\omega_0 t\, dt, \quad n = 0, 1, \ldots\ldots\ldots\ldots$$

Damped Systems due to a Simple Harmonic Exciting Force with Constant Amplitude

If $F(t)$ is non-periodic excitation

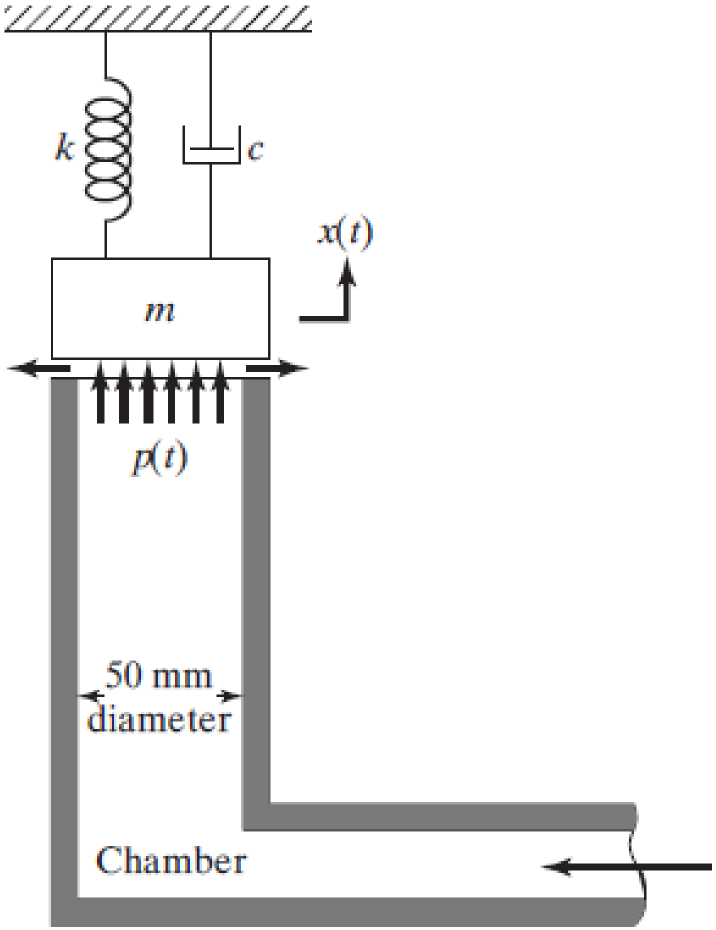


Step Load | Ramp Load | Rectangular Pulse Load

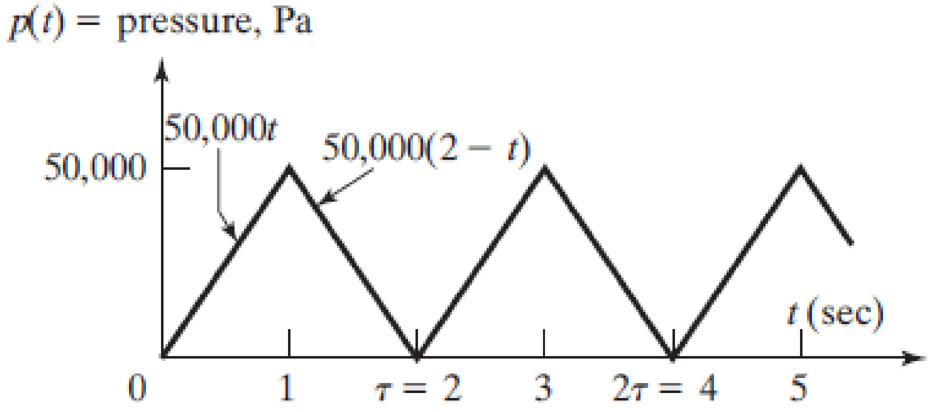
Triangular Pulse | Half-sine pulse load | Explosion load

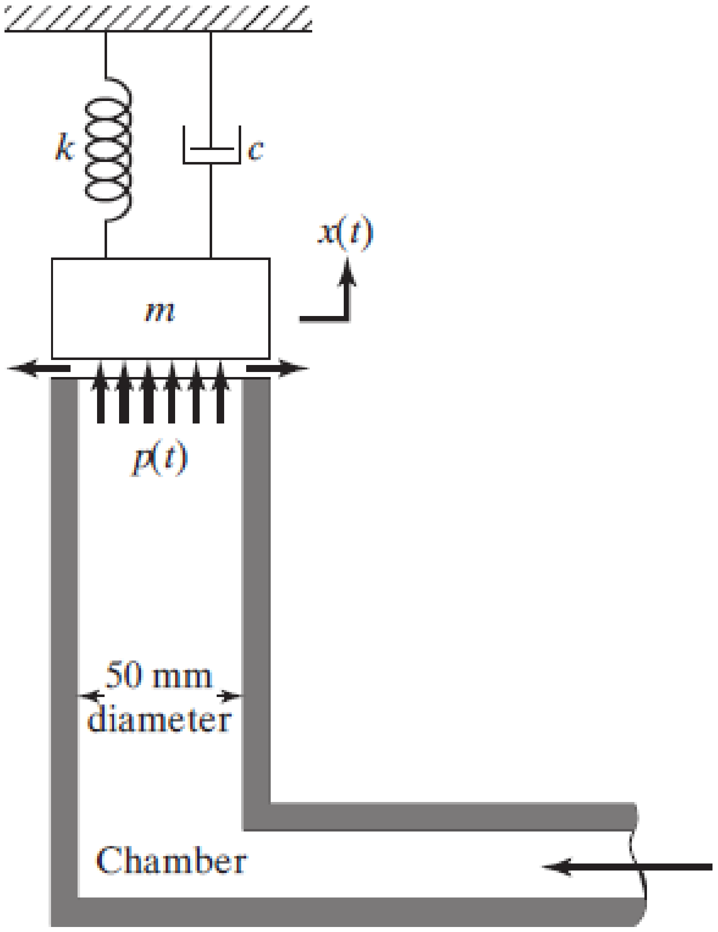
Solved Example:1 (Damped System due to Simple Harmonic Exciting Force with Constant Amplitude (non-sinusoidal))

In the study of vibrations of valves used in hydraulic control systems, the valve and its elastic stem are modeled as a damped spring-mass system, as shown in the following figure. In addition to the spring force and damping force, there is a fluid-pressure force on the valve that changes with the amount of opening or closing of the valve. Find the steady-state response of the valve when the pressure in the chamber varies as indicated in the figure. Assume ***k*** = 2500 N/m, ***c*** = 10 N-s/m, and ***m*** = 0.25 kg.



Solved Example:1 (Damped System due to Simple Harmonic Exciting Force with Constant Amplitude (non-sinusoidal))





Solved Example:1 (Damped System due to Simple Harmonic Exciting Force with Constant Amplitude (non-sinusoidal))

The forcing function can be expressed as

$$f(t) = A\,p(t)$$

where A is the cross-sectional area of the chamber, given by

$$A = 0.25\pi(50)^2 = 625\times10^{-3}\,\pi\ m^2$$

Since $p(t)$ is periodic with period $T = 2$ seconds and A is a constant, $F(t)$ is also a periodic function of period $T = 2$ seconds. The frequency of the forcing function is $\omega_0 = 2\pi/T = \pi\ rad/s$. $F(t)$ can be expressed in a Fourier series as

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)$$

Solved Example:1 (Damped System due to Simple Harmonic Exciting Force with Constant Amplitude (non-sinusoidal))

Since the function $F(t)$ is given by

$$F(t) = \begin{cases} 50000\,At & \text{for } 0 \le t \le \dfrac{T}{2} \\ 50000\,A(2-t) & \text{for } \dfrac{T}{2} \le t \le T \end{cases}$$

The Fourier coefficients a_n and b_n can be computed as:

$$a_0 = \frac{2}{2}\left[\int_0^1 50000\,At\,dt + \int_1^2 50000\,A(2-t)\,dt\right] = 50000\,A\,,$$

$$a_1 = \frac{2}{2}\left[\int_0^1 50000\,At\cos\pi t\,dt + \int_1^2 50000\,A(2-t)\cos\pi t\,dt\right] = -\frac{2\times10^5\,A}{\pi^2}$$

$$a_2 = \frac{2}{2}\left[\int_0^1 50000\,At\cos2\pi t\,dt + \int_1^2 50000\,A(2-t)\cos2\pi t\,dt\right] = 0$$

$$a_3 = \frac{2}{2}\left[\int_0^1 50000\,At\cos3\pi t\,dt + \int_1^2 50000\,A(2-t)\cos3\pi t\,dt\right] = -\frac{2\times10^5\,A}{9\pi^2}$$

$$b_1 = \frac{2}{2}\left[\int_0^1 50000\,At\sin\pi t\,dt + \int_1^2 50000\,A(2-t)\sin\pi t\,dt\right] = 0\,,$$

$$b_2 = \frac{2}{2}\left[\int_0^1 50000\,At\sin2\pi t\,dt + \int_1^2 50000\,A(2-t)\sin2\pi t\,dt\right] = 0$$

$$b_3 = \frac{2}{2}\left[\int_0^1 50000\,At\sin3\pi t\,dt + \int_1^2 50000\,A(2-t)\sin3\pi t\,dt\right] = 0$$

Solved Example:1 (Damped System due to Simple Harmonic Exciting Force with Constant Amplitude (non-sinusoidal))

Likewise, we can obtain $a_4 = a_6 = \ldots.. = b_4 = b_5 = b_6 = \ldots. = 0$. By considering only the first three harmonics, the forcing function can be approximated:

$$F(t) \approx 25000A - \frac{2\times10^5\ A}{\pi^2}\cos\omega_0 t - \frac{2\times10^5\ A}{9\pi^2}\cos 3\omega_0 t$$

From Eq. (4.8)

$$X = \frac{F_0/k}{\sqrt{\left[1-(\omega_0/\omega_n)^2\right]^2 + \left[2(c/c_{crit})(\omega_0/\omega_n)\right]^2}}$$

$$\phi = \tan^{-1}\left[\frac{\left[2(c/c_{crit})(\omega_0/\omega_n)\right]}{1-(\omega_0/\omega_n)^2}\right]$$

So,

$$x(t) = \frac{25000A}{k} - \frac{\left[2\times10^5 A/(k\pi^2)\right]}{\sqrt{\left[1-(\omega_0/\omega_n)^2\right]^2 + \left[2\zeta(\omega_0/\omega_n)\right]^2}}\cos(\omega_0 t - \phi_1)$$

$$- \frac{\left[2\times10^5 A/(9k\pi^2)\right]}{\sqrt{\left[1-9(\omega_0/\omega_n)^2\right]^2 + \left[6\zeta(\omega_0/\omega_n)\right]^2}}\cos(3\omega_0 t - \phi_3)$$

The natural frequency of the valve is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{0.25}} = 100 \text{ rad/s}$$

Solved Example:1 (Damped System due to Simple Harmonic Exciting Force with Constant Amplitude (non-sinusoidal))

the forcing frequency by

$$\omega_0 = \frac{2\pi}{T} = \pi \ rad/s$$

and the damping ratio:

$$\zeta = \frac{c}{c_{crit}} = \frac{c}{2\sqrt{km}} = \frac{10}{2\sqrt{(2500)(0.25)}} = 0.2$$

Thus the frequency ratio can be obtained:

$$\frac{\omega_0}{\omega_n} = \frac{\pi}{100} = 0.031416$$

The phase angles ϕ_1 and ϕ_3 can be computed as follows:

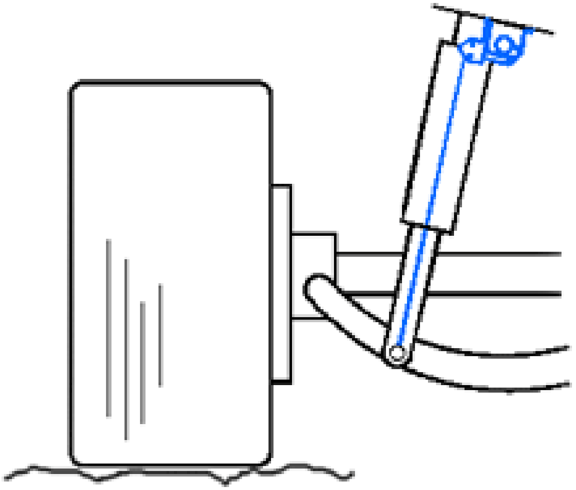
$$\phi_1 = \tan^{-1}\left[\frac{[2\times 0.2\times 0.031416]}{1-(0.031416)^2}\right] = 0.0125664 \ rad$$

$$\phi_3 = \tan^{-1}\left[\frac{[6\times 0.2\times 0.031416]}{1-\left(9(0.031416)^2\right)}\right] = 0.0380483 \ rad$$

The solution can be written as

$$x(t) = 0.019635 - 0.015930\cos(\pi t - 0.0125664) - 0.0017828\cos(3\pi t - 0.0380483)$$

Damped Systems Supported on a Foundation Subjected to Harmonic Vibration



Damped Systems Supported on a Foundation Subjected to Harmonic Vibration

The equation of motion is

$$m\ddot{x}_2 = c(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1)$$

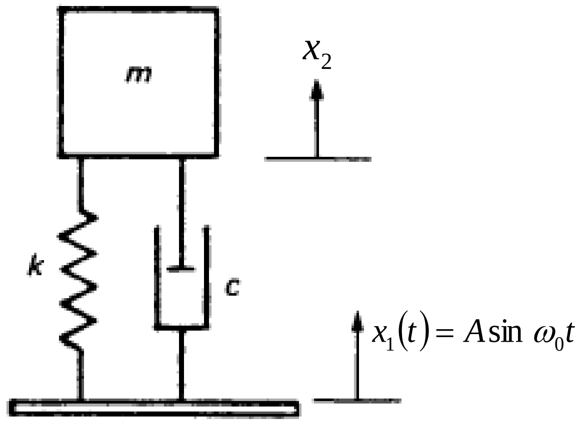
Rearranging

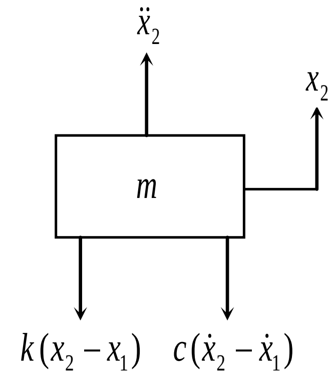
$$m\ddot{x}_2 + c\dot{x}_2 + kx_2 = c\dot{x}_1 + kx_1$$

If we are given

$$x_1(t) = X_1 \sin \omega_0 t$$

The equation becomes





$$m\ddot{x}_2 + c\dot{x}_2 + kx_2 = X_1(k \sin \omega_0 t + c\omega_0 \cos \omega_0 t)$$
$$= C \sin(\omega_0 t - \gamma)$$

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega_0 t$$

Where $\gamma = \tan^{-1} - \frac{c\omega_0}{k} = \tan^{-1}\left(-2\zeta \frac{\omega_0}{\omega_n}\right)$ and $C = X_1\sqrt{k^2 + (c\omega_0)^2}$. This shows that giving excitation to the base is equivalent to applying a harmonic force of magnitude C to the mass.

Damped Systems Supported on a Foundation Subjected to Harmonic Vibration

$$x_2 = X_1 \sqrt{\frac{k^2 + (c\omega_0)^2}{(k - m\omega_0^2)^2 + (c\omega_0)^2}} \sin(\omega_0 t - \phi_1 - \gamma)$$

Where

$$\phi_1 = \tan^{-1}\left(\frac{c\omega_0}{k - m\omega_0^2}\right)$$

Using trigonometric identities,

$$x_2 = X_2 \sin(\omega_0 t - \phi)$$

where X_2 and ϕ are given by

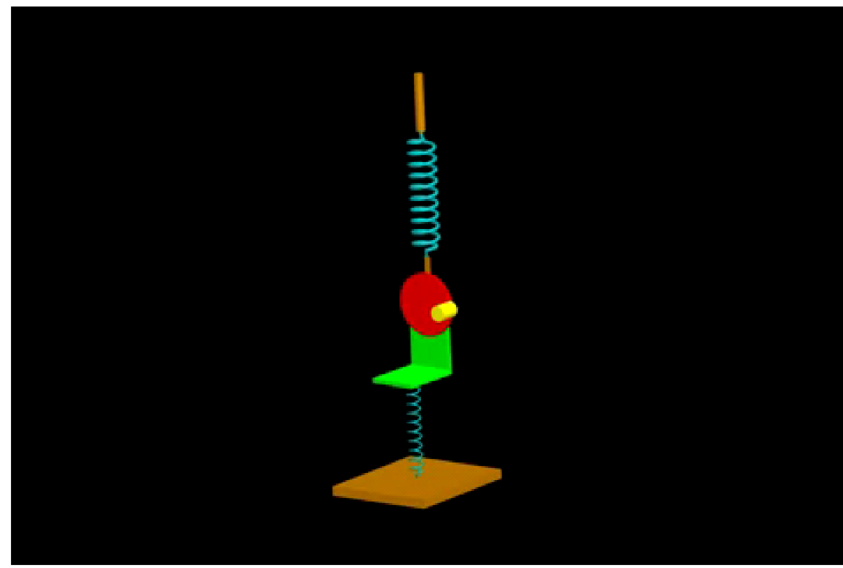
Damped Systems Supported on a Foundation Subjected to Harmonic Vibration

$$\frac{X_2}{X_1} = \sqrt{\frac{k^2 + (c\omega_0)^2}{(k - m\omega_0^2)^2 + (c\omega_0)^2}} = \sqrt{\frac{1 + \left(2\zeta\frac{\omega_0}{\omega_n}\right)^2}{\left[1 - \left(\frac{\omega_0}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega_0}{\omega_n}\right)^2}}$$

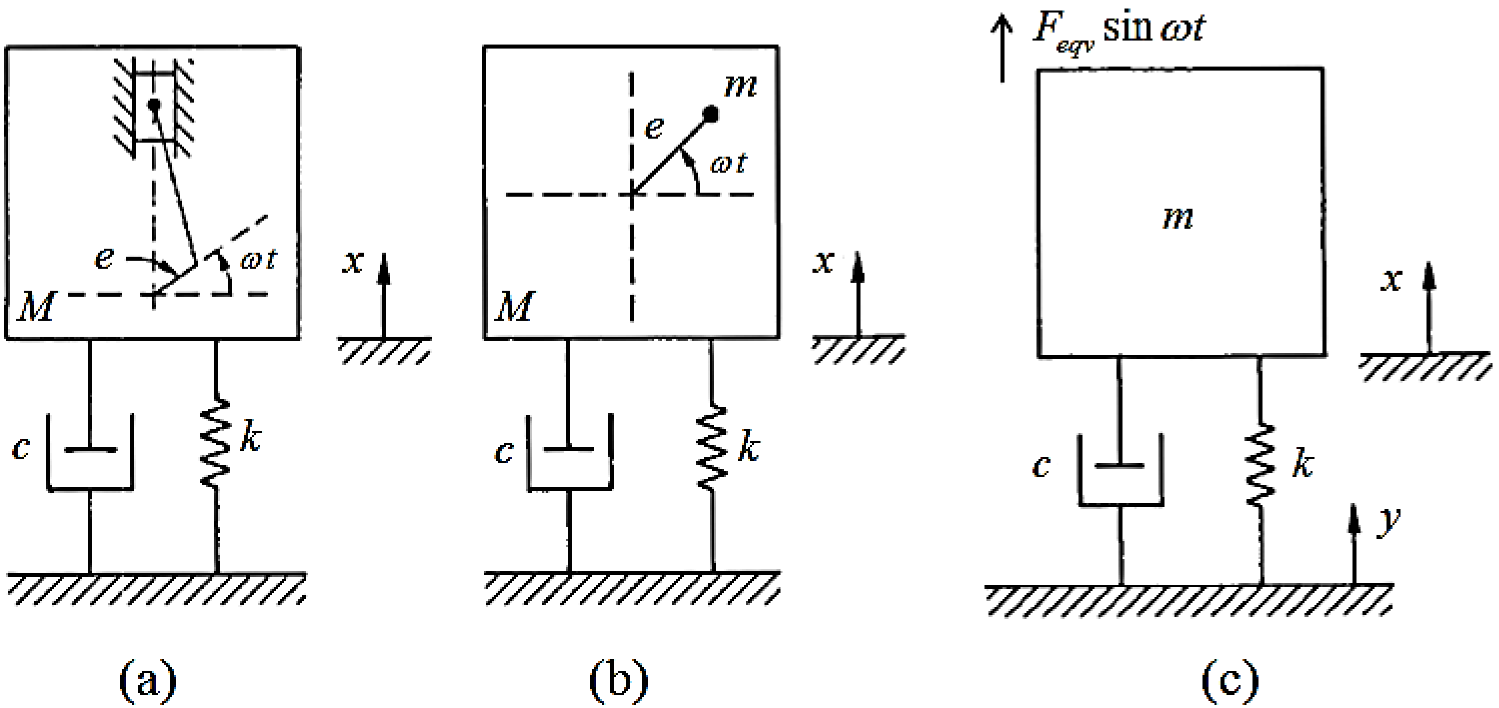
Where $\frac{X_2}{X_1}$ is called amplitude ratio.

$$\phi = \tan^{-1}\left[\frac{mc\omega_0^3}{k(k - m\omega_0^2) + (c\omega_0)^2}\right] = \tan^{-1}\left[\frac{2\zeta\left(\frac{\omega_0}{\omega_n}\right)^3}{1 + \left[(4\zeta^2 - 1)\left(\frac{\omega_0}{\omega_n}\right)^2\right]}\right]$$

Damped Forced Vibration due to Reciprocating or Rotating Unbalanced Masses



Damped Forced Vibration due to Reciprocating or Rotating Unbalanced Masses



(a) reciprocating motion of a piston, (b) rotation of an unbalanced rotor (c) equivalent model.

Damped Forced Vibration due to Reciprocating or Rotating Unbalanced Masses

Let

m = eccentric mass

M = mass of the machine including m

ω = angular velocity of rotation

If we assume it to be of SDOF model,

The vertical displacement is $x + e\sin\omega t$

The displacement of mass $(M - m)$ is $x(t)$

In each case the equation of motion is given by

$$-kx - c\dot{x} = (M - m)\ddot{x} + m\frac{d^2}{dt^2}(x + e\sin\omega t)$$

$$= (M - m)\ddot{x} + m\left(-\omega^2 e\sin\omega t + \ddot{x}\right)$$

$$= M\ddot{x} - me\omega^2\sin\omega t$$

Damped Forced Vibration due to Reciprocating or Rotating Unbalanced Masses

or $M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin \omega t$

or $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \dfrac{me\omega^2}{M} \sin \omega t$

where $me\omega^2$ is equivalent force of centrifugal force and $me\omega^2 \sin \omega t$ is vertical component.

$$x(t) = \frac{me\omega^2}{\sqrt{\left(k - M\omega^2\right)^2 + (c\omega)^2}} \sin(\omega t - \phi)$$

$$X = \frac{me\omega^2}{\sqrt{\left(k - M\omega^2\right)^2 + (c\omega)^2}}$$

$$\phi = \tan^{-1} \frac{c\omega}{k - M\omega^2}$$

By defining $\zeta = c/c_{crit}$ and $c_{crit} = 2M\omega_n$

$$\frac{MX}{me} = \frac{(\omega/\omega_n)^2}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta(\omega/\omega_n)\right]^2}}$$

$$\phi = \tan^{-1}\left[\frac{\left[2\zeta(\omega/\omega_n)\right]}{1 - (\omega/\omega_n)^2}\right]$$

Damped Forced Vibration due to Reciprocating or Rotating Unbalanced Masses

The maximum of $\frac{MX}{me}$ occurs when

$$\frac{d}{dFR}\left(\frac{MX}{me}\right)=0, \quad where\ FR\ \ frequancy\ ratio=\omega/\omega_n$$

The solution of this equation gives

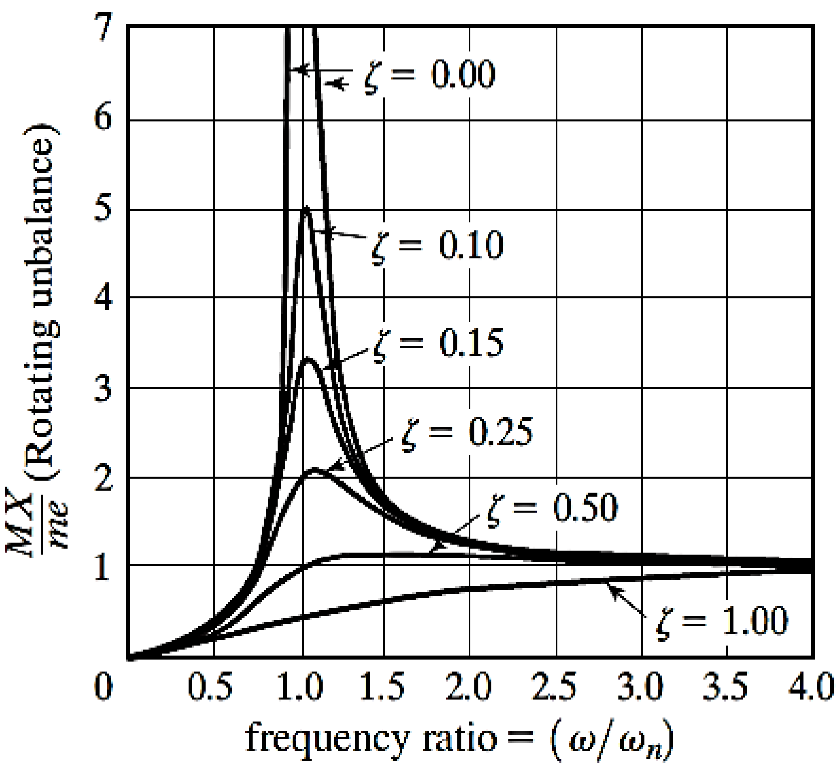
$$\frac{\omega}{\omega_n}=\frac{1}{\sqrt{1-2\zeta^2}}>1$$

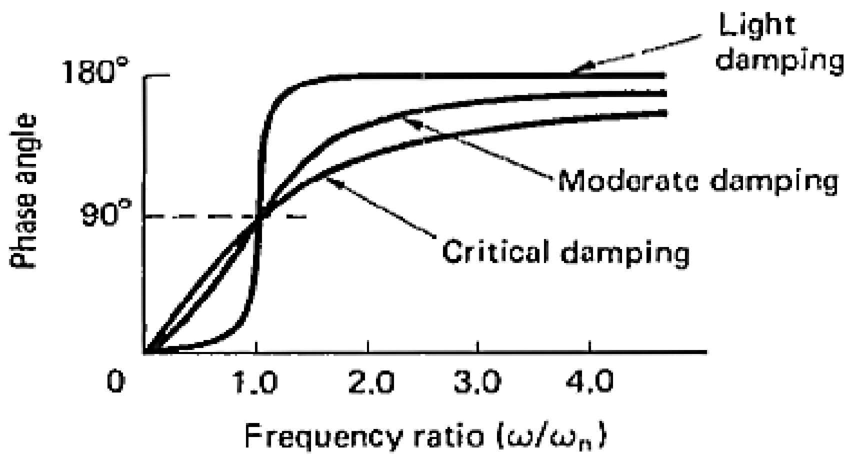
with the corresponding maximum value of $\frac{MX}{me}$ given by

$$\left(\frac{MX}{me}\right)_{\max}=\frac{1}{2\zeta\sqrt{1-2\zeta^2}}$$

This is the peak deflection when the frequency ratio varies from resonance and occurs to the right of the resonance value of $\omega/\omega_n=1$

Damped Forced Vibration due to Reciprocating or Rotating Unbalanced Masses






QUIZ

Quiz

A slider-crank mechanism is used to impart motion to the base of a spring-mass-damper system, as shown in Figure (P3.1). Approximating the base motion $y(t)$ as a series of harmonic functions, drive the equation of motion and find the response (natural frequency, damping ratio) of the mass for $m = 1$ kg, $c = 10$ N-s/m, $k = 100$ N/m, $r = 10$ cm, $l = 1$ m, and $\omega = 100$ rad/s.

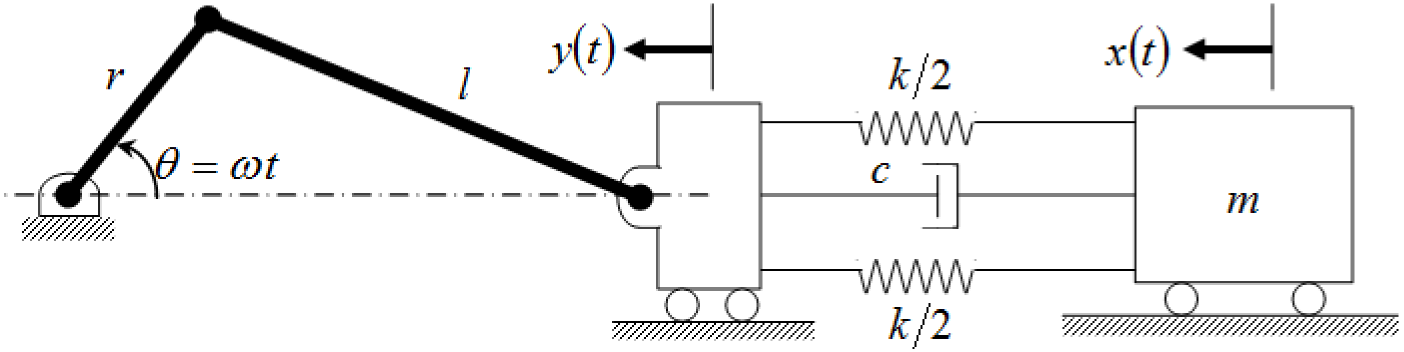


Fig. P3.1

Thank you