Rayleigh's Methods of Dimensional Analysis

Step 1: Setting up the equation

Write the equation in terms of dimensions only, i.e. replace the quantities with their respective units. The equation then becomes

$$D = C\,l^a \rho^b \mu^c V^d g^e \Longrightarrow \frac{ML}{T^2} = (L)^a \left(\frac{M}{L^3}\right)^b \left(\frac{M}{LT}\right)^c \left(\frac{L}{T}\right)^d \left(\frac{L}{T^2}\right)^e$$

On the left side, we have $M^1\,L^1\,T^{-2}$, which is equal to the dimensions on the right side. Therefore, the exponents of the right side must be such that the units are $M^1\,L^1\,T^{-2}$

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Step 2: Solving for the exponents

Equate the exponents to each other in terms of their respective fundamental units:

M: $1 = b + c$ since $M^1 = M^b M^c$

L: $1 = a - 3b - c + d + e$ since $L^1 = L^a L^{-3b} L^{-c} L^d L^e$

T: $-2 = -c - d - 2e$ since $T^{-2} = T^{-c} T^{-d} T^{-2e}$

$$\frac{ML}{T^2} = (L)^a \left(\frac{M}{L^3}\right)^b \left(\frac{M}{LT}\right)^c \left(\frac{L}{T}\right)^d \left(\frac{L}{T^2}\right)^e$$

It is seen that there are three equations, but 5 unknown variables. This means that a complete solution cannot be obtained. Thus, we choose to solve a, b, and d in terms of c and e. These choices are based on experience. Therefore

From M: $b = 1 - c$ (i)

From T: $d = 2 - c - 2e$ (ii)

From L: $a = 1 + 3b + c - d - e$ (iii)

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Step 2: Solving for the exponents

Solving (i), (ii), and (iii) simultaneously, we obtain

a = 2 - c + e

Substituting the exponents back into the original equation, we obtain

$$D = C l^{(2+e-c)} \rho^{(1-c)} \mu^{c} V^{(2-c-2e)} g^{e}$$

Which means

$$D = C\, l^{2}\, l^{e}\, l^{-c} \rho^{1} \rho^{-c} \mu^{c}\, V^{2} V^{-c}\, V^{2e}\, g^{e}$$

Collecting like exponents together,

$$D = C \left(\frac{V^2}{lg}\right)^{-e} \left(\frac{Vl\rho}{\mu}\right)^{-c} \rho l^2 V^2$$

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Step 3: Determining the dimensionless groups

Note that e and c are unknown. Consider the following cases:

If e = 1 then (iv) becomes $\left(\frac{lg}{V^2}\right)$

If e = -1 then (iv) becomes $\left(\frac{V^2}{lg}\right)$

If c = 1 then (v) becomes $\left(\frac{\mu}{l\rho V}\right)$

If c = -1 then (v) becomes $\left(\frac{l\rho V}{\mu}\right) = \left(\frac{lV}{\nu}\right)$

Where ν is the kinematic viscosity of the fluid.

And so on for different exponents. It turns out that:

$$Reynolds\ Number \equiv \frac{Vl}{\nu} = N_R = Re \qquad Froude\ Number \equiv \left(\frac{V^2}{lg}\right)^{\frac{1}{2}} = \frac{V}{\sqrt{lg}} = N_F = Fr$$

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Example: The resisting force **R** of a supersonic plane during flight can be considered as dependent upon the length of the aircraft **L**, velocity **V**, air viscosity **μ**, air density , and bulk modulus of air **k**. Express the functional relationship between the variables and the resisting force.

Solution:

$$R = f(l, V, \mu, \rho, K) \Rightarrow R = Al^a, V^b, \mu^c, \rho^d, K^e \quad (1)$$

Where: A = Non dimensional constant

Substituting the powers on both sides of the equation

$$MLT^{-2} = AL^a(LT^{-1})^b(ML^{-1}T^{-1})^c(ML^{-3})^d(ML^{-1}T^{-2})^e$$

Equating the powers of MLT on both sides

Power of M $\Rightarrow \quad 1 = c + d + e$

Power of L $\Rightarrow \quad 1 = a + b - c - 3d - e$

Power of T $\Rightarrow \quad -2 = -b - c - 2e$

Rayleigh's Methods of Dimensional Analysis

Since the unkown(5) are more than number of equations(3). So expressing a, b & c in terms of d & e

$$d = 1 - c - e$$

$$b = 2 - c - 2e$$

$$a = 1 - b + c + 3d + e = 1 - (2 - c - 2e) + c + 3(1 - c - e) + e$$

$$= 1 - 2 + c + 2e + c + 3 - 3c - 3e + e = 2 - c$$

Substituting the values in (1), we get

$$R = A l^{2-c} V^{2-c-2e} \mu^c \rho^{1-c-e} K^e = A l^2 V^2 \rho (l^{-c} V^{-c} \mu^c \rho^{-c})(V^{-2e} \rho^{-e} K^e)$$

$$R = A \rho l^2 V^2 \left[\left(\frac{\mu}{\rho V l} \right)^c \left(\frac{K}{\rho V^2} \right)^e \right]$$

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Example: The velocity of propagation of a pressure wave through a liquid can be expected to depend on the elasticity of the liquid represented by the bulk modulus K, and its mass density ρ. Establish by D. A. the form of the possible relationship.

Assume: $u = C\,K^{a}\,\rho^{b}$

U = velocity $= L\,T^{-1}$, $\rho = M\,L^{-3}$, $K = M\,L^{-1}\,T^{-2}$

$L\,T^{-1} = M^{a}\,L^{-a}\,T^{-2a} \times M^{b}\,L^{-3b}$

M: $0 = a + b$

L: $1 = -a - 3b \;\Rightarrow\; a = 0.5,\; b = -0.5 \;\Rightarrow\; u = C\sqrt{\dfrac{K}{\rho}}$

T: $-1 = -2a$

Rayleigh's Methods of Dimensional Analysis

Rayleigh Method has the following limitations:

1. The premise that an exponential relationship exists between the variables.
2. Rayleigh method Becomes laborious if variables are more than fundamental dimensions (MLT)
3. Rayleigh method doesn't provide any information regarding number of dimensionless groups to be obtained as a result of dimension analysis.
4. Rayleigh method is not always so straightforward. Consider the situation of flow over a U-notched weir.
5. When a large number of variables are involved, Rayleigh's method becomes lengthy.

Buckingham's - Π Theorem/Method of Dimensional Analysis

Say that we have m number of variables (e.g. 6 variables, which are D, l, ρ, μ, V, and g) and n number of dimensional quantities (e.g. 3 dimensional quantities , which are M, L, and T).

In general we can derive (m-n) independent dimensionless groups, such as Re and Fr, often denoted $\pi_{1,}\ \pi_{2,}\ \pi_{3,}\ \ldots\ldots\ \pi_{m-n}$

Buckingham's - Π Theorem/Method of Dimensional Analysis

Using the following steps:

1. Write down the dimensions for all variables $X_1, X_2 \ldots X_m$
2. Select n of the variables - say $Y_1, Y_2 \ldots Y_n$. These are called the repeating variables, leaving the remaining variables as $R_1, R_2 \ldots R_{m-n}$. Note that if the analysis does not work out, we could always go back and repeat using new RVs. These repeating variables will appear in all the π terms. Note that there are certain restrictions on our choice :
   - None of the repeating variables can be dimensionless
   - No two repeating variables can have the same overall dimension. For instance, D, the pipe diameter, and ε, the roughness height, both have dimension of L, and so cannot both be used as repeating variables.

Buckingham's - Π Theorem/Method of Dimensional Analysis

3. Form the terms or dimensionless π groups. We can find the combination by dimensional analysis, by writing the group in the form

$$\pi_1 = Y_1^{a_{11}} Y_2^{a_{12}} \ldots\ldots Y_n^{a_{1n}} R_1$$

$$\pi_2 = Y_1^{a_{21}} Y_2^{a_{22}} \ldots\ldots Y_n^{a_{2n}} R_2$$

$$\pi_3 = Y_1^{a_{31}} Y_2^{a_{32}} \ldots\ldots Y_n^{a_{3n}} R_3$$

.

.

$$\pi_n = Y_1^{a_{n1}} Y_2^{a_{n2}} \ldots\ldots Y_n^{a_{nn}} R_{m-n}$$

Which are all dimensionless quantities, i.e. having units of $M^0 L^0 T^0$

4. Determine π groups; we find that the exponents **a**

5. Having worked out all the dimensionless groups, the relationship between the variables can be expressed as a relationship between the various groups. Typically we can write this as one group (for example 1 as a function of the others $\Pi_1 = f(\Pi_2, \Pi_3, \ldots)$

Buckingham's - Π Theorem/Method of Dimensional Analysis

1. Now, this method will be illustrated by the same example as that for Rayleigh method, the drag on a ship. $D = C l^a \rho^b \mu^c V^d g^e$ ⟹ $\frac{ML}{T^2} = (L)^a \left(\frac{M}{L^3}\right)^b \left(\frac{M}{LT}\right)^c \left(\frac{L}{T}\right)^d \left(\frac{L}{T^2}\right)^e$

   we have 6 variables, which are D, l, ρ, μ, V, and g and 3 dimensions, which are M, L, and T.

   Where m = 6, n = 3, so there will be m - n = 3 π groups.

2. We will select ρ, V, and l as the repeating variables (RV), leaving the remaining quantities as D, μ, and g.

3. Form the terms or dimensionless π groups

$$\pi_1 = \rho^{a_{11}} V^{a_{12}} l^{a_{13}} D$$

$$\pi_2 = \rho^{a_{21}} V^{a_{22}} l^{a_{23}} \mu$$

$$\pi_3 = \rho^{a_{31}} V^{a_{32}} l^{a_{33}} g$$

Buckingham's - Π Theorem/Method of Dimensional Analysis

4. Determine π groups; we find that the exponents **a**

$$\pi_1 \quad M^0L^0T^0 = \left(\frac{M}{L^3}\right)^{a_{11}}\left(\frac{L}{T}\right)^{a_{12}}(L)^{a_{13}}\left(\frac{ML}{T^2}\right)$$

$$\pi_2 \quad M^0L^0T^0 = \left(\frac{M}{L^3}\right)^{a_{21}}\left(\frac{L}{T}\right)^{a_{22}}(L)^{a_{23}}\left(\frac{M}{LT}\right)$$

$$\pi_3 \quad M^0L^0T^0 = \left(\frac{M}{L^3}\right)^{a_{31}}\left(\frac{L}{T}\right)^{a_{32}}(L)^{a_{33}}\left(\frac{L}{T^2}\right)$$

$$a_{11} = -1 \quad a_{12} = -2 \quad a_{13} = -2$$

$$a_{21} = -1 \quad a_{22} = -1 \quad a_{23} = -1$$

$$a_{31} = 0 \quad a_{32} = -2 \quad a_{33} = 1$$

$$\pi_1 = \rho^{-1}V^{-2}l^{-2}D = \frac{D}{\rho V^2 l^2}$$

$$\pi_2 = \rho^{-1}V^{-1}l^{-1}\mu = \frac{\mu}{\rho V l} = \frac{\nu}{Vl}$$

$$\pi_3 = \rho^0 V^{-2} l g = \frac{lg}{V^2}$$

5. $\Pi_1 = f(\Pi_2, \Pi_3, \ldots)$

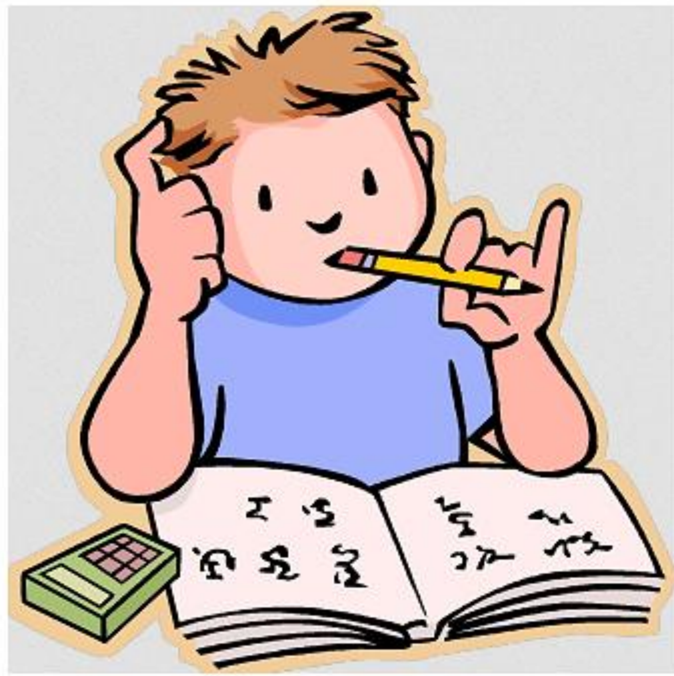
$$\pi_1 = f(\pi_2, \pi_3)$$

$$\frac{D}{\rho V^2 l^2} = f\left(\frac{\mu}{\rho V l}, \frac{lg}{V^2}\right) \Rightarrow D = \rho V^2 l^2 f\left(\frac{\mu}{\rho V l}, \frac{lg}{V^2}\right)$$

Buckingham's - Π Theorem/Method of Dimensional Analysis

1. Note that this is the same result as obtained with the Rayleigh Method, but with the Buckingham π Method, we did not have to assume a functional dependence

2. Both Buckingham's method and Rayleigh's method of dimensional analysis determine only the relevant independent dimensionless parameters of a problem, but not the exact relationship between them.



Homework

HW(3)

When freewheeling, the angular velocity Ω of a windmill is found to be a function of the windmill diameter D, the wind velocity V, the air density ρ, the windmill height H as compared to the atmospheric boundary layer height L, and the number of blades N:

$$\Omega = \mathrm{fcn}\left(D, V, \rho, \frac{H}{L}, N\right)$$

Viscosity effects are negligible. Find appropriate pi groups for this problem and rewrite the function in dimensionless form.

Thank you