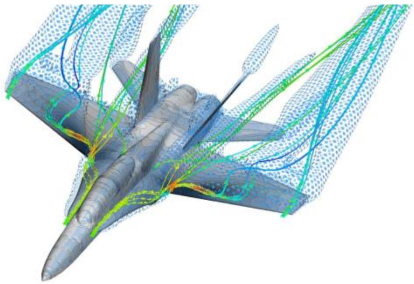
بسم الله الرحمن الرحيم

Fluid Mechanics I



Fayoum University

Faculty of Engineering
Mechanical Engineering Dept.

Lecture (2)

on

Dimensional Analysis and Similarity

By

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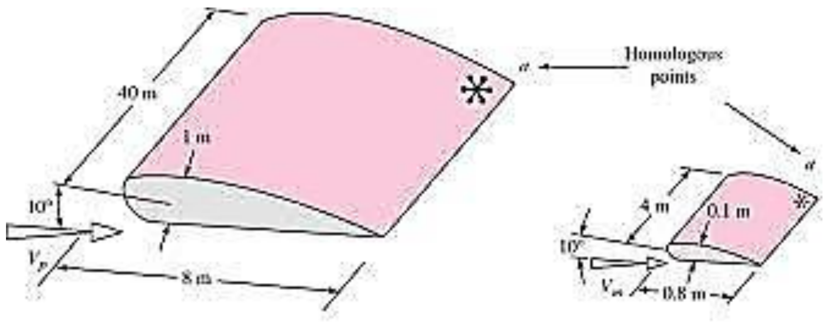
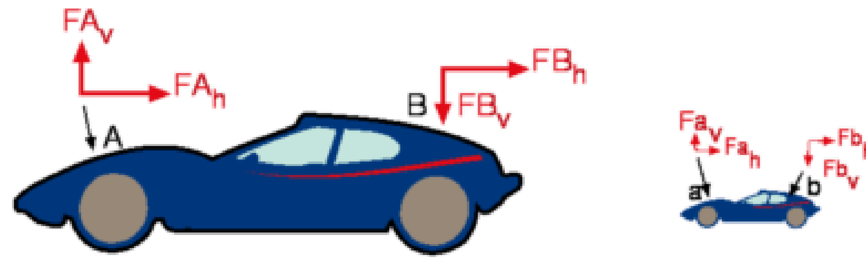
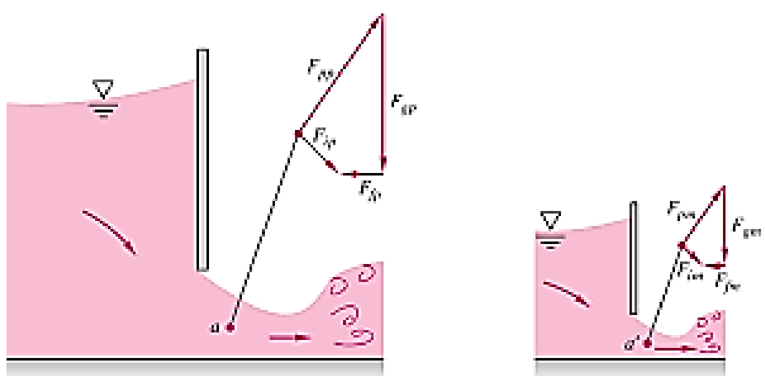
Purposes of Dimensional Analysis



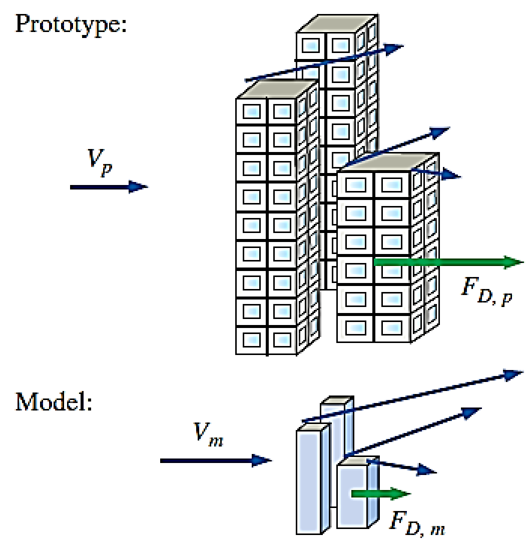
Prototype   Model



Prototype   Model



Prototype   Model



Dimensional Analysis Concept

Before discussing the technique of dimensional analysis, we first explain the underlying concept of dimensional analysis; the principle of similarity.

There are three necessary conditions for complete similarity between a model and a prototype:

Geometric similarity: the model must be the same shape as the prototype, but may be scaled by some constant scale factor.

$$\frac{L_p}{L_m} = \frac{B_p}{B_m} = \frac{D_p}{D_m} = L_r$$

- Where: L_p, B_p and D_p are Length, Breadth, and diameter of prototype and L_m, B_m, D_m are Length, Breadth, and diameter of model.
- Lr= Scale ratio

Dimensional Analysis Concept

Kinematic similarity: which means that the velocity at any point in the model flow must be proportional (by a constant scale factor) to the velocity at the corresponding point in the prototype flow. Specifically, for kinematic similarity the velocity at corresponding points must scale in magnitude and must point in the same relative direction. You may think of geometric similarity as length-scale equivalence and kinematic similarity as time-scale equivalence. Geometric similarity is a prerequisite for kinematic similarity.

$$\frac{V_{p1}}{V_{m1}} = \frac{V_{p2}}{V_{m2}} = V_r; \quad \frac{a_{p1}}{a_{m1}} = \frac{a_{p2}}{a_{m2}} = a_r$$

- Where: V_{p1} & V_{p2} and a_{p1} & a_{p2} are velocity and accelerations at point 1 & 2 in prototype and V_{m1} & V_{m2} and a_{m1} & a_{m2} are velocity and accelerations at point 1 & 2 in model.
- V_r and a_r are the velocity ratio and acceleration ratio

Dimensional Analysis Concept

Dynamic similarity: Dynamic similarity is achieved when all forces in the model flow scale by a constant factor to corresponding forces in the prototype flow (force-scale equivalence). As with geometric and kinematic similarity, the scale factor for forces can be less than, equal to, or greater than one.

$$\frac{(F_i)_p}{(F_i)_m} = \frac{(F_v)_p}{(F_v)_m} = \frac{(F_g)_p}{(F_g)_m} = F_r$$

- Where: $(F_i)_p$, $(F_v)_p$ and $(F_g)_p$ are inertia, viscous and gravitational forces in prototype and $(F_i)_m$, $(F_v)_m$ and $(F_g)_m$ are inertia, viscous and gravitational forces in model.
- F_r is the Force ratio

Dimensional Analysis Concept

- Kinematic similarity is a necessary but insufficient condition for dynamic similarity.
- It is thus possible for a model flow and a prototype flow to achieve both geometric and kinematic similarity, yet not dynamic similarity.
- All three similarity conditions must exist for complete similarity to be ensured.
- In a general flow field, complete similarity between a model and prototype is achieved only when there is geometric, kinematic, and dynamic similarity.

Dimensional Analysis Concept

Characteristics		Unit (SI)	Dimension (MLT)	Dimension (FLT)
Geometry	Length	m	L	
	Area	m^2	L^2	
	Volume	m^3	L^3	
Kinematic	Time	S	T	
	Velocity	m/S	L/T	
	Acceleration	m/S^2	L/T^2	
	Discharge	m^3/S	L^3/T	
Dynamic	Mass	Kg	M	$FL^{-1}T^2$
	Force	N ($Kg\text{-}m/S^2$)	MLT^{-2}	F
	Pressure	Pa (N/m^2)	$ML^{-1}T^{-2}$	FL^{-2}
	Energy	J (N-m)	ML^2T^{-2}	FL
	Power	Watt (N-m/S)	ML^2T^{-3}	FLT^{-1}

Types of Forces Encountered in Fluid Phenomenon

Inertia Force, Fi: It is equal to product of mass and acceleration in the flowing fluid.

Viscous Force, Fv: It is equal to the product of shear stress due to viscosity and surface area of flow.

Gravity Force, Fg: It is equal to product of mass and acceleration due to gravity.

Pressure Force, Fp: it is equal to product of pressure intensity and cross-sectional area of flowing fluid.

Surface Tension Force, Fs: It is equal to product of surface tension and length of surface of flowing fluid.

Elastic Force, Fe: It is equal to product of elastic stress and area of flowing fluid.

Dimensionless Numbers

Dimensionless numbers are the numbers which are obtained by dividing the inertia force by viscous force or gravity force or pressure force or surface tension force or elastic force.

As this is ratio of once force to other, it will be a dimensionless number. These are also called non-dimensional parameters.

The following are most important dimensionless numbers.

- Reynold's Number
- Froude's Number
- Euler's Number
- Weber's Number
- Mach's Number

Dimensionless Numbers

Eulers's Number, Eu: It is the ratio of inertia force to the pressure force of flowing fluid.

$$E_u = \sqrt{\frac{Fi}{Fp}} = \sqrt{\frac{Mass.\dfrac{Velocity}{Time}}{Pressure.\,Area}} = \sqrt{\frac{\rho\dfrac{Volume}{Time}.Velocity}{Pressure.\,Area}}$$

$$= \sqrt{\frac{\rho Q.V}{P.A}} = \sqrt{\frac{\rho A V.V}{P.A}} = \sqrt{\frac{V^2}{P/\rho}} = \frac{V}{\sqrt{P/\rho}}$$

- **Weber's Number, We:** It is the ratio of inertia force to the surface tension force of flowing fluid.

$$We = \sqrt{\frac{Fi}{Fg}} = \sqrt{\frac{Mass.\dfrac{Velocity}{Time}}{Surface\,Tension\,per.\,Length}} = \sqrt{\frac{\rho\dfrac{Volume}{Time}.Velocity}{Surface\,Tension\,per.\,Length}}$$

$$= \sqrt{\frac{\rho Q.V}{\sigma.L}} = \sqrt{\frac{\rho A V.V}{\sigma.L}} = \sqrt{\frac{\rho L^2 V^2}{\sigma.L}} = \frac{V}{\sqrt{\dfrac{\sigma}{\rho L}}}$$

Dimensionless Numbers

Mach's Number, M: It is the ratio of inertia force to the elastic force of flowing fluid.

$$M=\sqrt{\frac{Fi}{Fe}}=\sqrt{\frac{Mass.\dfrac{Velocity}{Time}}{Elastic\ Stress.\ Area}}=\sqrt{\frac{\rho\dfrac{Volume}{Time}.Velocity}{Elastic\ Stress.\ Area}}$$

$$=\sqrt{\frac{\rho Q.V}{K.A}}=\sqrt{\frac{\rho AV.V}{K.A}}=\sqrt{\frac{\rho L^2V^2}{KL^2}}=\frac{V}{\sqrt{K/\rho}}=\frac{V}{C}$$

$$Where: C=\sqrt{K/\rho}$$

Model Laws or Similarity Laws

We have already learned that for dynamic similarity, ratio of corresponding forces acting on prototype and model should be equal i.e.

$$\frac{(F_v)_p}{(F_v)_m} = \frac{(F_g)_p}{(F_g)_m} = \frac{(F_p)_p}{(F_p)_m} = \frac{(F_s)_p}{(F_s)_m} = \frac{(F_e)_p}{(F_e)_m} = \frac{(F_I)_p}{(F_I)_m}$$

- Force of inertial comes in play when sum of all other forces is not equal to zero which mean:

$$\left(F_v + F_g + F_p + F_s + F_e\right) = \left(F_I\right)$$

Thus dynamic similarity require that

$$\frac{\left(F_v + F_g + F_p + F_s + F_e\right)_p}{\left(F_v + F_g + F_p + F_s + F_e\right)_m} = \frac{(F_I)_p}{(F_I)_m}$$

- In case all the forces are equally important, the above two equations cannot be satisfied for model analysis

Model Laws or Similarity Laws

However, for practical problems it is seen that one force is most significant compared to others and is called predominant force or most significant force.

Thus, for practical problem only the most significant force is considered for dynamic similarity. Hence, models are designed on the basis of ratio of force, which is dominating in the phenomenon.

Finally, the laws on which models are designed for dynamic similarity are called models laws or laws of similarity. The followings are these laws

- Reynold's Model Law
- Froude's Model Law
- Euler's Model Law
- Weber's Model Law
- Mach's Model Law

Reynold's Model Law

It is based on Reynold's number and states that Reynold's number for model must be equal to the Reynolds number for prototype.

Reynolds Model Law is used in problems where viscous forces are dominant. These problems include:

- Pipe Flow
- Resistance experienced by submarines, airplanes, fully immersed bodies etc

$$(\mathrm{Re})_P = (\mathrm{Re})_m \quad or \quad \frac{V_P L_P}{\upsilon_P} = \frac{V_m L_m}{\upsilon_m}$$

$$\frac{V_P L_P}{V_m L_m \left(\dfrac{\upsilon_P}{\upsilon_m}\right)} = \frac{V_r L_r}{\upsilon_r} = 1$$

$$where: V_r = \frac{V_P}{V_m}, L_r = \frac{L_P}{L_m}, \upsilon_r = \frac{\upsilon_P}{\upsilon_m}$$

Reynold's Model Law

The Various Ratios for Reynolds's Law are obtained as

$$\sin ce \left(\frac{VL}{\upsilon}\right)_P = \left(\frac{VL}{\upsilon}\right)_m \quad and \quad \upsilon = \mu / \rho$$

$$\text{Velocity Ratio: } V_r = \frac{V_P}{V_m} = \frac{L_m}{L_P}\frac{\upsilon_P}{\upsilon_m} = \frac{\upsilon_r}{L_r}$$

$$\text{Time Ratio: } Tr = \frac{T_P}{T_m} = \frac{L_P/V_P}{L_m/V_m} = \frac{L_r}{V_r}$$

$$\text{Acceleration Ratio: } a_r = \frac{a_P}{a_m} = \frac{V_P/T_P}{V_m/T_m} = \frac{Vr}{Tr}$$

$$\text{Discharge Ratio: } Q_r = \frac{A_P V_P}{A_m V_m} = L_r^2 V_r$$

$$\text{Force Ratio: } F_r = m_r a_r = \rho_r Q_r V_r = \rho_r L_r^2 V_r V_r = \rho_r L_r^2 V_r^2$$

$$\text{Power Ratio: } P_r = F_r . V_r = \rho_r L_r^2 V_r^2 V_r = \rho_r L_r^2 V_r^3$$

Reynold's Model Law-Ex:1

Q. A pipe of diameter *1.5 m* is required to transport an oil of specific gravity *0.90* and viscosity *3x10⁻²* *poise* at the rate of *3000litre/s.* Tests were conducted on a *15 cm* diameter pipe using water at 20°C. Find the velocity and rate of flow in the model.

Solution:

Prototype Data:

- Diameter, D_p= 1.5m
- Viscosity of fluid, μ_p= 3×10^{-2} poise
- Discharge, Q_p =3000litre/sec
- Sp. Gr., S_p=0.9
- Density of oil=ρ_p=0.9x1000 =900kg/m^3

Model Data:

- Diameter, Dm=15cm =0.15 m
- Viscosity of water, μ_m =1×10^{-2} poise
- Density of water, ρ_m=1000kg/m^3
- Velocity of flow V_m=?
- Discharge Q_m=?

For pipe flow,

According to Reynolds' Model Law

$$\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho_p V_p D_p}{\mu_p} \Rightarrow \frac{V_m}{V_p} = \frac{\rho_p D_p}{\rho_m D_m}\frac{\mu_m}{\mu_p}$$

$$\frac{V_m}{V_p} = \frac{900\times1.5}{1000\times0.15}\frac{1\times10^{-2}}{3\times10^{-2}} = 3.0$$

$$\text{Since } V_p = \frac{Q_p}{A_p} = \frac{3.0}{\pi/4(1.5)^2}$$

$$= 1.697 m/s$$

$$\therefore V_m = 3.0V_p = 5.091 m/s$$

$$and\ Q_m = V_m A_m = 5.091\times\pi/4(0.15)^2$$

$$= 0.0899 m^3/s$$

Reynold's Model Law-Ex:2

Q. A ship 300m long moves in sea water, whose density is 1030 kg/m^3. A 1:100 model of this ship is to be tested in a wind tunnel. The velocity of air in the wind tunnel around the model is 30m/s and the resistance of the model is 60N.

Determine the velocity of ship in sea water and also the resistance of ship in sea water. The density of air is given as 1.24kg/m^3. Take the kinematic viscosity of air and sea water as 0.012 stokes and 0.018 stokes respectively.

Solution:

For Prototype

- Length, L_p= 300m
- Fluid = sea water
- Density of sea water, ρ_p= 1030 kg/m^3
- Kinematic Viscosity, v_p=0.018 stokes
  $=0.018 \times 10^{-4}$ m^2/s
- Let Velocity of ship, V_p
- Resistance, F_p

- For Model
  - Scale ratio = L_p/L_m=100
  - Length, L_m= L_p/100 = 3m
  - Fluid = air
  - Density of air, ρ_m= 1.24 kg/m^3
  - Kinematic Viscosity, v_m=0.012 stokes
    $=0.012 \times 10^{-4}$ m^2/s
  - Velocity of ship, V_m=30 m/s
  - Resistance, F_m = 60 N

Reynold's Model Law-Ex:2

For dynamic similarity between model and prototype, the Reynolds number for both of them should be equal.

$$\left(\frac{VL}{\upsilon}\right)_p = \left(\frac{VL}{\upsilon}\right)_m \Rightarrow V_p = \frac{\upsilon_p}{\upsilon_m}\frac{L_m}{L_p}V_m$$

$$Vp = \frac{0.018 \times 10^{-4}}{0.012 \times 10^{-4}}\frac{3}{300}30 = 0.2 m/s$$

$$Since \quad \text{Resistance} = \text{Mass} \times \text{Acceleration} = \rho L^2 V^2$$

$$Thus \quad \frac{F_p}{F_m} = \frac{\left(\rho L^2 V^2\right)_p}{\left(\rho L^2 V^2\right)_m} = \frac{1030}{1.24}\left(\frac{300}{3}\right)^2\left(\frac{0.2}{30}\right)^2 = 369.17$$

$$F_p = 369.17 \times 60 = 22150.2 N$$

Froude's Model Law

It is based on Froude's number and states that Froude's number for model must be equal to the Froude's number for prototype.

Froude's Model Law is used in problems where gravity forces is only dominant to control flow in addition to inertia force. These problems include:

- Free surface flows such as flow over spillways, weirs, sluices, channels etc.
- Flow of jet from orifice or nozzle
- Waves on surface of fluid
- Motion of fluids with different viscosities over one another

$$(Fe)_P = (Fe)_m \; or \; \frac{V_P}{\sqrt{g_P L_P}} = \frac{V_m}{\sqrt{g_m L_m}} \; or \; \frac{V_P}{\sqrt{L_P}} = \frac{V_m}{\sqrt{L_m}}$$

$$\frac{V_P}{V_m\left(\sqrt{\dfrac{L_P}{L_m}}\right)} = V_r / \sqrt{L_r} = 1; \; where: V_r = \frac{V_P}{V_m}, L_r = \frac{L_P}{L_m}$$

Froude's Model Law

The Various Ratios for Reynolds's Law are obtained as;

$$\sin ce \frac{V_P}{\sqrt{L_P}} = \frac{V_m}{\sqrt{L_m}}$$

$$\text{Velocity Ratio: } V_r = \frac{V_P}{V_m} = \sqrt{\frac{L_p}{L_m}} = \sqrt{L_r}$$

$$\text{Time Ratio: } Tr = \frac{T_P}{T_m} = \frac{L_P/V_P}{L_m/V_m} = \frac{L_r}{\sqrt{L_r}} = \sqrt{L_r}$$

$$\text{Acceleration Ratio: } a_r = \frac{a_P}{a_m} = \frac{V_P/T_P}{V_m/T_m} = \frac{Vr}{Tr} = \frac{\sqrt{L_r}}{\sqrt{L_r}} = 1$$

$$\text{Discharge Ratio: } Q_r = \frac{A_P V_P}{A_m V_m} = L_r^2 V_r = L_r^2 \sqrt{L_r} = L_r^{5/2}$$

$$\text{Force Ratio: } Fr = m_r a_r = \rho_r Q_r V_r = \rho_r L_r^2 V_r V_r = \rho_r L_r^2 V_r^2 = \rho_r L_r^2 L_r = \rho_r L_r^3$$

$$\text{Power Ratio: } Pr = Fr.Vr = \rho_r L_r^2 V_r^2 V_r = \rho_r L_r^2 V_r^3 = \rho_r L_r^2 \left(\sqrt{L_r}\right)^3 = \rho_r L_r^{7/2}$$

Froude's Model Law-Ex:3

In the In the model test of a spillway the discharge and velocity of flow over the model were 2 m^3/s and 1.5 m/s respectively. Calculate the velocity and discharge over the prototype which is 36 times the model size.

Solution: Given that

For Model

- Discharge over model, Q_m=2 m^3/sec
- Velocity over model, V_m = 1.5 m/sec
- Linear Scale ratio, L_r =36

For Prototype

- Discharge over prototype, Q_p =?
- Velocity over prototype V_p=?

For Discharge

$$\frac{Q_p}{Q_m} = (L_r)^{2.5} = (36)^{2.5}$$

$$Q_p = (36)^{2.5} \times 2 = 15552\ m^3/\sec$$

For Dynamic Similarity,

Froude Model Law is used

$$\frac{V_p}{V_m} = \sqrt{L_r} = \sqrt{36} = 6$$

$$V_p = 6 \times 1.5 = 9\ m/\sec$$

Froude's Model LawLaw-Ex:4

The characteristics of the spillway are to be studied by means of a geometrically similar model constructed to a scale of 1:10.

1. If 28.3 m^3/s, is the maximum rate of flow in prototype, what will be the corresponding flow in model?
2. If 2.4 m/s, 50mm and 3.5 Nm are values of velocity at a point on the spillway, height of hydraulic jump and energy dissipated per second in model, what will be the corresponding velocity, height of hydraulic jump and energy dissipation per second in prototype?

Froude's Model Law-Ex:4

Solution: Given that

For Model

- Discharge over model, Q_m=?
- Velocity over model, V_m = 2.4 m/sec
- Height of hydraulic jump, H_m =50 mm
- Energy dissipation per second, E_m =3.5 Nm
- Linear Scale ratio, L_r =10

For Prototype

- Discharge over model, Q_p=28.3 m^3/sec
- Velocity over model, V_p =?
- Height of hydraulic jump, H_p =?
- Energy dissipation per second, E_p =?

Froude's Model Law -Ex:4

For Discharge:

$$\frac{Q_p}{Q_m} = L_r^{2.5} = 10^{2.5}$$

$$Q_m = 28.3/10^{2.5} = 0.0895 \quad m^3/\sec$$

For Velocity:

$$\frac{V_p}{V_m} = \sqrt{L_r} = \sqrt{10}$$

$$V_p = 2.4 \times \sqrt{10} = 7.589 \quad m/\sec$$

For Hydraulic Jump:

$$\frac{H_p}{H_m} = L_r = 10$$

$$H_p = 50 \times 10 = 500 \ mm$$

For Energy Dissipation:

$$\frac{E_p}{E_m} = L_r^{3.5} = 10^{3.5}$$

$$E_p = 3.5 \times 10^{3.5} = 11067.9 \quad Nm/\sec$$

Classification of Models

Undistorted or True Models: are those which are geometrically similar to prototype or in other words if the scale ratio for linear dimensions of the model and its prototype is same, the models is called undistorted model. The behavior of prototype can be easily predicted from the results of undistorted or true model.

Distorted Models: A model is said to be distorted if it is not geometrically similar to its prototype. For distorted models different scale ratios for linear dimension are used.

For example, if for the river, both horizontal and vertical scale ratio are taken to be same, then depth of water in the model of river will be very very small which may not be measured accurately.

- **The followings are the advantages of distorted models**
  - The vertical dimension of the model can be accurately measured
  - The cost of the model can be reduced
  - Turbulent flow in the model can be maintained
- Though there are some advantage of distorted models, however the results of such models cannot be directly transferred to prototype.

Classification of Models

Scale Ratios for Distorted Models

Let: $(L_r)_H = \frac{L_P}{L_m} = \frac{B_P}{B_m}$ Scale ratio for horizontal direction

$(L_r)_V = \frac{h_P}{h_m}$ =Scale ratio for vertical direction

Scale Ratio for Velocity: $V_r = V_P / V_m = \frac{\sqrt{2gh_P}}{\sqrt{2gh_m}} = \sqrt{(L_r)_V}$

Scale Ratio for area of flow: $A_r = A_P / A_m = \frac{B_P h_P}{B_m h_m} = (L_r)_H (L_r)_V$

Scale Ratio for discharge: $Q_r = Q_P / Q_m = \frac{A_P V_P}{A_m V_m} = (L_r)_H (L_r)_V \sqrt{(L_r)_V} = (L_r)_H (L_r)_V^{3/2}$

Classification of Models-Ex:5

The discharge through a weir is 1.5 m^3/s. Find the discharge through the model of weir if the horizontal dimensions of the model=1/50 the horizontal dimension of prototype and vertical dimension of model =1/10 the vertical dimension of prototype.

Solution:

Discharge of River= $Q_p = 1.5 m^3/s$

Scale ratio for horizontal direction= $(L_r)_H = \frac{L_P}{L_m} = 50$

Scale ratio for vertical direction= $(L_r)_V = \frac{h_P}{h_m} = 10$

Since Scale Ratio for discharge: $Qr = Q_P / Q_m = (L_r)_H (L_r)_V^{3/2}$

$$\therefore Q_p / Q_m = 50 \times 10^{3/2} = 1581.14$$

$$\Rightarrow Q_m = 1.5 / 1581.14 = 0.000948 \quad m^3 / s$$

Classification of Models-Ex:6

A river model is to be constructed to a vertical scale of 1:50 and a horizontal of 1:200. At the design flood discharge of 450 m^3/s, the average width and depth of flow are 60 m and 4.2 m respectively. Determine the corresponding discharge in model and check the Reynolds' Number of the model flow.

$$Discharge\ of\ River = Q_p = 450 m^3/s$$

$$Width = B_p = 60 m\ and\ Depth = y_p = 4.2\ m$$

$$\text{Horizontal scale ratio}=(L_r)_H = \frac{B_P}{B_m} = 200$$

$$\text{Vertical scale ratio}=(L_r)_V = \frac{y_P}{y_m} = 50$$

$$\text{Since Scale Ratio for discharge: } Q_r = Q_P / Q_m = (L_r)_H (L_r)_V^{3/2}$$

$$\therefore Q_p / Q_m = 200 \times 50^{3/2} = 70710.7$$

$$\Rightarrow Q_m = 450 / 70710.7 \quad 6.365 \times 10^{-3} \quad m^3/s$$

Classification of Models-Ex:6

$$\text{Reynolds Number, } Re_m = \left(\frac{VL}{\upsilon}\right)_m$$

$$L_m = 4R_m$$

$$Width = B_m = B_p / (L_r)_H = 60/200 = 0.3m$$

$$Depth = y_m = y_p / (L_r)_V = 4.2/50 = 0.084\ m$$

$$A_m = B_m y_m = 0.3 \times 0.084 = 0.0252m$$

$$P_m = B_m + 2y_m = 0.3 + 2 \times 0.084 = 0.468m$$

$$R = \frac{A_m}{P_m} = \frac{0.0252}{0.468} = 0.05385$$

$$\text{Kinematic Viscosity of water} = \upsilon = 1 \times 10^{-6} m^2/\sec$$

$$Re = \left(\frac{4VR}{\upsilon}\right)_m = \left(\frac{4 \times 0.253 \times 0.05385}{1 \times 10^{-6}}\right) = 54492.31$$

$$> 2000$$

$\therefore$ Flow is in turbulent range

Thank you